General Solutions
2003 Rice Math Tournament
February 22, 2003

1. **Answer: $\frac{2000}{3}$**

   Let m be the number of miles, then we want $250 + .1m = 150 + .25m$. We solve the equality and find that $m = \frac{2000}{3}$ miles.

2. **Answer: $\frac{19}{34}$**

   The probability of having two red socks or two blue socks are $\frac{12}{17} \cdot \frac{11}{16}$ and $\frac{5}{17} \cdot \frac{4}{16}$ respectively, totaling $\frac{19}{34}$.

3. **Answer: Sixth (last)**

   F finishes in fourth, fifth or sixth place in event 2. F cannot finish fifth as the athlete that does so also scores second in event 1. Thus if M finishes fourth, F must finish sixth.

4. **Answer: -3**

   If $a * b = 2$, then $a^2 + ab + 3b + 1 = 2$ and $b(a+3) = 1 - a^2$. And from this we see that if $a \neq -3$, then $b = (1-a^2)/(a+3)$ satisfies $a*b$. However, if $a = -3$, then for any b, $a*b = a^2+(a+3)b+1 = a^2+1 = 10$.

5. **Answer: 49**

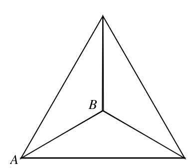
   There are 5 single-digit rotationally-symmetric numbers (0, 1, 2, 5, and 8) and 6 two-digit ones (11, 22, 55, 88, 69, and 96). Since three-digit rotationally-symmetric numbers are composed of a two-digit and a one-digit rotationally-symmetric number, there are $5 \cdot 6 = 30$. Four-digit rotationally-symmetric numbers are of the form $1AB1$ or 2002 since numbers greater than 2003 are not included. Noting that AB is rotationally-symmetric but can also be 00, we see that there are $6+1 = 7$ rotationally-symmetric numbers of the form $1AB1$, so there are 8 four-digit rotationally-symmetric numbers up to 2003. In total, there are therefore $5 + 6 + 30 + 8 = 49$ rotationally-symmetric numbers from 0 to 2003.

6. **Answer: 6**

   If Ani answers a single question wrong, then she can get at most 4 points, so we know that she can't miss any. From there, we see that the only way she can score 5 is if she answers 5 right and skips one. There are 6 choices for the one she skips, so there are 6 ways she can score a 5 on the exam.

7. **Answer: 7**

   To better visualize this, imagine we are looking down at the tetrahedron from above. (Note that it doesn't matter which vertices A and B are.)



   There is 1 path using one side, 2 paths using 2 sides, 2 paths using 3 sides, and 2 paths using 4 sides. (The only restriction was that she can't use any *edge* more than once, so it's okay to pass through A again.) This gives 7 ways total.

8. **Answer: $2 + 2\sqrt{2}$**

   Let ℓ be the length of a leg, and let h be the length of the hypotenuse. We can orient the triangle such that the base and height are both ℓ, so $\frac{1}{2}\ell^2 = 1$, which implies that $\ell = \sqrt{2}$. Then $h = \sqrt{\ell^2 + \ell^2} = \sqrt{4} = 2$. Hence, the perimeter is $2 + 2\sqrt{2}$.

9. **Answer: 13**

   The factors of 95 are 1, 5, 19, and 95 making 5/19/95 the only solution in 1995. The useful factors of 96 are 4, 6, 8, 12, 16, and 24, giving rise to the solutions 4/24/96, 6/16/96, 8/12/96, and 12/8/96. 97

is prime so 1997 has no Multiplication Days. The only useful factors of 98 are 7 and 14 giving 7/14/98. 1999 has two days, 9/11/99 and 11/9/99. 2000 has no solutions. 1/1/2001 is the only day in 2001. 2002 however has the two solutions 1/2/02 and 2/1/02. Lastly, 2003 has two days, 1/3/03 and 3/1/03. This gives us a total of 13 days.

10. **Answer: $\mathbf{8, 12}$**

    Let l, w be the length and width, respectively. Then, $2l + 2w = 40$ and $lw = 96$. Thus, $l(20 - l) = 96$, which implies $l^2 - 20l + 96 = 0$. This factors as $(l - 8)(l - 12) = 0$ giving $l = 8$ or $l = 12$. Whichever value l takes, w takes the other.

11. **Answer: 4096**

    This is simply $2^{2^{2^2}-2^2} = 2^{16-4} = 2^{12} = 4096$.

12. **Answer: $\mathbf{(4, 18, 8)}$**

    Let x, y, z be the number of bottles of commercial sprays X, Y and Z, respectively that we use. Then, we need $x + z = 12$, $2x + z = 16$ and $2x + y = 26$. Subtracting the first equation from the second yields $x = 4$. Hence we can find $y = 18$ and $z = 8$.

13. **Answer: $\mathbf{\frac{5}{26}}$**

    The possible cards that are six or less are sixes, fives, fours, threes and twos. There are two red suits, giving 10 desirable possibilities out of the 52-card deck. Thus, the probability is $\frac{10}{52} = \frac{5}{26}$.

14. **Answer: 126**

    If the hours is 1 or 10, we need the sum of the other digits to be 11. There are 4 possibilities: 29, 38, 47 and 56. If the hour is 2 or 11, we have 5 possibilities, and hours 3 and 12 give us 6 possibilities each. If the hour is 4, 5, 6, or 7, we get 6 possibilities, while 8 gives us 5 possibilities, and 9 gives us 4. Altogether, this gives us 63 different possibilities. Each readout occurs twice each day, so the answer is 126.

15. **Answer: 6**

    The scenario can be translated into three equations: $L = 4M + 2$, $B = 2(L - 11) - 3$, and $B + 15 = 7M$. Solving these yields the solution $M = 6, L = 26$, $B = 27$. Thus Maggie is 6 years old.

16. **Answer: 3**

    O only connects to N and U while W only connects to T. Thus, there are more than two towns between them. As N and T both connect to P, there are a minimum of three towns between towns O and W.

17. **Answer: 32**

    The area of the lower left triangle is $\frac{1}{2} \cdot 3 \cdot 8 = 12$, while the area of the upper right triangle is $\frac{1}{2} \cdot 2 \cdot 5 = 5$, and the area of the rectangle is $3 \cdot 5 = 15$. Summing these gives us 32.

    Notice that the entire enclosed region is *not* a triangle. The hypotenuse of the lower left triangle has a slope of $\frac{3}{8}$, while the hypotenuse of the upper right triangle has a slope of $\frac{2}{5}$.

18. **Answer: 10 meters**

    Let x be the width and y be the length of the corral. If the inner fence is as short as possible, then it is simply a straight line of length y. Then $2x + 3y = 100$ and $xy = 350$. We can solve one for x and substitute in to yield the quadratic equations $3y^2 - 100y + 700 = 0$. This factors as $(3y - 70)(y - 10) = 0$. Thus $y = 10$ or $\frac{70}{3}$. The shorter of these is 10 meters.

19. **Answer: $\mathbf{\frac{12}{25}}$**

    The possible odd sums are 3, 5, 7 and 9. There are 2 ways to get 3 and 9 and 4 ways to get 5 and 7. This gives us 12 desirable results out of the $5^2 = 25$ possible results. Thus we get a probability of $\frac{12}{25}$.

20. **Answer: 4**

    a and b are the roots of the polynomial $x^2 - kx + k$. (There are various ways to show this. One is to multiply out $(x - a)(x - b)$, another is to substitute $b = k - a$ into the equation $ab = k$.) Using the quadratic equation, this polynomial has roots $x = \frac{k \pm \sqrt{k^2 - 4k}}{2}$. These roots are real if and only if $k^2 - 4k \geq 0$. And given that $k > 0$, we find that this only holds if $k \geq 4$.

21. **Answer: 168**

The leading digit must be 1, 2, 3, 4, 5, 6 or 7. One of the three digits must be a 5. First assume that the first digit is 5. That leaves $9 \cdot 8$ choices for the last two digits, since repetition is not allowed. Now, if the second digit is 5, we have 6 possibilities for the first digit, which leaves 8 choices for the third, giving us $6 \cdot 8 = 48$ possibilities. Similarly, if the third digit is 5, there are 6 choices for the first and 8 for the second, giving us another 48 possibilities. Therefore, there are $9 \cdot 8 + 2 \cdot 48 = 168$ different numbers.

22. **Answer: 200**

Let x be the number of \$250 CD players in stock and y be the number of \$400 players. To maximize profit, he should sell as many units as possible, so $x + y = 250$. Meanwhile, $250x + 400y \leq 70000$ due to the inventory limit. From the first equation, we obtain $y = 250 - x$, and plugging this into the inequality yields $250x + 400(250 - x) \leq 70,000$, so $-150x \leq -30,000$ and $x \geq 200$. Since the merchant makes the most profit off of the \$400 model, he should sell as many of those as possible. So, he should sell as few of the \$250 model as he can, while still selling as many units as possible. Since $x \geq 200$, he should minimize x, and therefore stock 200 of the \$250 model.

23. **Answer: 420**

There are 7 letters and hence 7! orderings. However, the two A's are indistinguishable, hence swapping them leaves us with the same permutation. We have thus overcounted by a factor of 2. Likewise, there are $3! = 6$ rearrangements of the three E's, thus the answer is $\frac{7!}{2!3!} = 420$ permutations.

24. **Answer: $\frac{50}{9}$ liters $= 5\frac{5}{9}$ liters**

Initially, $.25 \cdot 50 = 12.5$ liters of the solution is pure brine. Suppose that we add x liters of the 75% solution to the tank. This adds another $.75x$ liters of pure brine, and we want the ratio of pure brine to the total volume of the solution to be .3. Therefore,

$$\begin{aligned} \frac{12.5 + .75x}{50 + x} &= .3 \\ 12.5 + .75x &= 15 + .3x \\ .45x &= 2.5 \\ x &= \frac{50}{9}. \end{aligned}$$

25. **Answer: $\frac{12}{49}$**

The four remaining pins can either be in a straight line down one of the sides, or in a diamond with pin 4 as its topmost point. If Peter hits pin x on his first roll and pin y on his second, we find that the following sequences knock over 6 pins:

$$(1,3), (2,3), (2,6), (3,1), (3,2), (3,3), (5,5), (5,6), (5,7), (6,5), (6,2), (7,5).$$

Thus, there are 12 ways for Peter to knock over 6 pins. There are 49 ways for him to bowl a frame, so his chance of knocking over 6 pins is $\frac{12}{49}$.

26. **Answer: $\frac{1}{x+1} + \frac{-1}{x+2}$**

The denominators must be the linear factors of the polynomial $x^2 + 3x + 2$, which are $x + 1$ and $x + 2$. Thus, we want to find A and B such that

$$\begin{aligned} \frac{1}{x^2 + 3x + 2} &= \frac{A}{x+1} + \frac{B}{x+2} \\ \frac{1}{x^2 + 3x + 2} &= \frac{A(x+2) + B(x+1)}{x^2 + 3x + 2} \\ \frac{1}{x^2 + 3x + 2} &= \frac{(A+B)x + (2A+B)}{x^2 + 3x + 2}. \end{aligned}$$

Setting the numerators equal, we obtain the system of equations

$$\begin{aligned} A + B &= 0 \\ 2A + B &= 1. \end{aligned}$$

Solving this, we find that $A = 1$ and $B = -1$, so $\frac{1}{x^2+3x+2} = \frac{1}{x+1} - \frac{1}{x+2}$.

27. **Answer: 2003**

Let N be the number of coins. Since one coin is leftover when the loot is divided between 13 pirates, $N-1$ must be divisible by 13. Similarly, we find that $N-1$ must be divisible by 11 and 7 as well.

Since 7, 11, and 13 are all prime numbers, and $N > 1$, we see that $N-1$ must be a multiple of $7 \cdot 11 \cdot 13 = 1001$. However, $N-1$ cannot be 1001 since this would give $N = 1002$, which is even. Therefore, $N-1$ must be 2002, so $N = 2003$.

28. **Answer: 119**

Pick one of the 17 vertices. It has diagonals with 14 other vertices (not counting the vertex nor its neighbors). However, each diagonal is then counted twice, once for each end. This gives $\frac{17 \cdot 14}{2} = 119$ diagonals.

29. **Answer: $\mathbf{(250,000 : 125,000 : 125,000)}$**

Let c, b, and s be the amount invested in CDs, bonds and stocks, respectively. He invests one-fourth of his money in stocks, so $s = \frac{1}{4} \cdot 500,000 = 125,000$. The remaining \$375,000 is divided between CDs and bonds, so $c + b = 375000$, and thus $c = 375,000 - b$. The total return he wants is 10%, so

$$\begin{aligned}
.1c + .08b + .12s &= .1 \cdot 500,000 \\
.1 \cdot (375,000 - b) + .08b + .12 \cdot 125,000 &= 50,000 \\
37,500 - .1b + .08b + 15,000 &= 50,000 \\
.02b &= 2,500 \\
b &= 125,000.
\end{aligned}$$

It then follows that $c = 250,000$, and the ordered triple $(c : b : s)$ is $(250,000 : 125,000 : 125,000)$.

30. **Answer: $\mathbf{20\sqrt{3}}$**

Let the length, width and height be l, w, and h respectively. Then $wl = 15$, $wh = 8$, and $lh = 10$. Multiplying these three equations together yields $l^2w^2h^2 = 8 \cdot 15 \cdot 10 = 1200$. Thus, volume is $lwh = \sqrt{1200} = 20\sqrt{3}$.

31. **Answer: 346**

Let w be the number of white and r be the number of red. Then $r + w < 30$ and and $w < 5r$. For $r = 1$ there are 4 choices for w. For $r = 2$ there are 9 choices, $r = 3$ gives 14 choices, $r = 4$ gives 19 choices, and $r = 5$ gives 24 choices. Therefore for $r \leq 5$, there are $4 + 9 + 14 + 19 + 24 = 70$ possible pairs (r, w).

When $6 \leq r \leq 28$, the restriction $w < 5r$ is overridden by $w + r < 30$. So for each such r, there are $29 - r$ possible choices for w. This gives $23 + 22 + 21 + \cdots + 2 + 1 = \frac{23 \cdot 24}{2} = 276$ more possible pairs (r, w). Hence, there are $276 + 70 = 346$ possibilities in all.

32. **Answer: 1445**

First observe that we can discard the fifth condition as it is subsumed by the first two. The four conditions admit $4 \cdot 5 \cdot 5 \cdot 5 = 500$, $5 \cdot 5 \cdot 5 \cdot 5 = 625$, $\binom{10}{4} = 210$, and $\binom{9}{4} = 126$ numbers respectively. The number of four-digit numbers that satisfy at least two conditions is $3\binom{5}{4} + \binom{4}{4} = 16$. Noting that no three conditions can hold simultaneously, we know by inclusion-exclusion that the total number of four-digit numbers satisfying any of the conditions is $500 + 625 + 210 + 126 - 16 = 1445$.

33. **Answer: $\mathbf{\frac{8 \cdot 6 \cdot 4}{3!} = 32}$**

Let's pick the vertices of a triangle in order. We have 8 choices for the first vertex. The second vertex cannot be the first one we chose, nor can it be the one directly opposite that vertex on the octagon (since that diagonal wasn't drawn in). Hence, there are only 6 choices for the second vertex. The third vertex cannot be either of the first two vertices, nor can it be either of the two vertices opposite these two. Thus, there are 4 choices for the third vertex. Altogether, this gives us $8 \cdot 6 \cdot 4$ ways of choosing a triangle.

However, each unique triangle is counted 3! times since this is the number of ways of ordering its vertices. So in the end, there are $\frac{8 \cdot 6 \cdot 4}{3!} = 32$ triangles.