

If the K frame is the rest frame of a chosen particle, then time intervals $d\tau$ measured in it are referred to as the particle's **proper time** determination. Then the intervals measured in all other frames are given by the time dilation formula

$$d\tau = dt \left/ \sqrt{1 - \frac{v^2}{c^2}} \right. , \quad (7)$$

and the scalar spacetime quantity

$$dx_\mu dx^\mu \equiv c^2 d\tau^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2) \quad (8)$$

is an invariant. For light, which cannot possess a rest frame, we have $d\tau = 0$ as in Eq. (1). For particles, which are “**time-like**,” we have $d\tau > 0$.

1.1 Velocity Transformations

Now consider particles moving with respect to the two frames, measured with a velocity $\mathbf{u}$ in K , and with $\mathbf{u}'$ in K' . The inverse Lorentz transformation can be written with differentials:

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$$\begin{aligned} dx &= \gamma(dx' + v dt') , & dy &= dy' \\ dz &= dz' , & dt &= \gamma\left(dt' + \frac{v}{c^2} dx'\right) . \end{aligned} \quad (9)$$

Dividing through by the time differential, we develop the **Lorentz velocity transformation** relations:

$$\begin{aligned} u_x &= \frac{dx}{dt} = \frac{u'_x + v}{1 + vu'_x/c^2} , \\ u_y &= \frac{u'_y}{\gamma(1 + vu'_x/c^2)} , \\ u_z &= \frac{u'_z}{\gamma(1 + vu'_x/c^2)} . \end{aligned} \quad (10)$$

These reduce to the familiar Galilean velocity transformations when $v \ll c$.

Plot: Lorentz transformation of velocities.

- Observe that for $u'_x \rightarrow c$, the transformation yields $u_x \rightarrow c$ also, implying constancy of the speed of light as viewed in all inertial frames.

* Note also that the velocity components in the y and z directions are altered by the transformation because of time dilation.

For arbitrary boosts $\mathbf{v}$, a rotation of coordinates yields the simple replacements

$$u_x \rightarrow u_{\parallel} \quad \text{and} \quad \sqrt{u_y^2 + u_z^2} \rightarrow u_{\perp} \quad , \quad (11)$$

i.e. in terms of components parallel to ($\parallel$) and perpendicular to ($\perp$) the boost $\mathbf{v}$. Then, the directions of the velocities in the two frames are related by the **aberration formula**:

$$\tan \theta = \frac{u_{\perp}}{u_{\parallel}} = \frac{u' \sin \theta'}{\gamma(u' \cos \theta' + v)} \quad \text{for} \quad u' = |\mathbf{u}'| \quad . \quad (12)$$

Since $v \geq 0$ and $\gamma \geq 1$, this *collapses the angle towards the direction of the boost*. The azimuthal angle ϕ remains unaltered in the Lorentz transformation. In the particular case of $u' \rightarrow c$,

$$\tan \theta = \frac{\sin \theta'}{\gamma(\cos \theta' + v/c)} \quad \Rightarrow \quad \cos \theta = \frac{\cos \theta' + v/c}{1 + (v/c) \cos \theta'} \quad , \quad (13)$$

which describes the **aberration of light**.

- In the radiation aberration formula, if we set $\theta' = \pi/2$, to identify a general transverse direction, then

$$\tan \theta \approx \frac{c}{\gamma v} \quad \Rightarrow \quad \sin \theta \approx \frac{1}{\gamma} \quad . \quad (14)$$

This defines the effect angular extent of the “aberration beam” in the K frame, so that if a charge is moving relativistically, even though it may radiate quasi-isotropically in its own reference frame, to the stationary observer, the emission appears mostly within a **Lorentz cone** of opening half-angle $\theta \sim 1/\gamma$ when $\gamma \gg 1$. Very few photons will be emerge with $\theta \gtrsim 1/\gamma$. This collimation phenomenon is termed a **relativistic beaming** effect.

1.2 The Doppler Effect

Now look at how transmitted signals differ in their appearance when received by a moving observer. Consider a source of light that emits the wave at a frequency $\omega' = \omega_0$ in the K' frame. The source is stationary at the origin of the K' frame, which moves with speed v along the x -axis of the K frame. The period of the wave in the K' frame is $2\pi/\omega'$, and time dilation yields a period in the K frame of

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$$\Delta t = \frac{2\pi\gamma}{\omega'} . \quad (15)$$

After a period has elapsed, the next oscillation of the wave is emitted at a position offset from the first oscillation by a distance $l = v\Delta t$ along x .

Plot: Geometry for the Doppler effect.

For an observer at angle θ with respect to the x -axis, the projected distance shortening to the second point of emission is $d = v\Delta t \cos \theta$. The difference in arrival times Δt_A between radiation emitted at points 1 and 2 is

$$\Delta t_A = \Delta t - \frac{d}{c} = \Delta t \left(1 - \frac{v}{c} \cos \theta \right) . \quad (16)$$

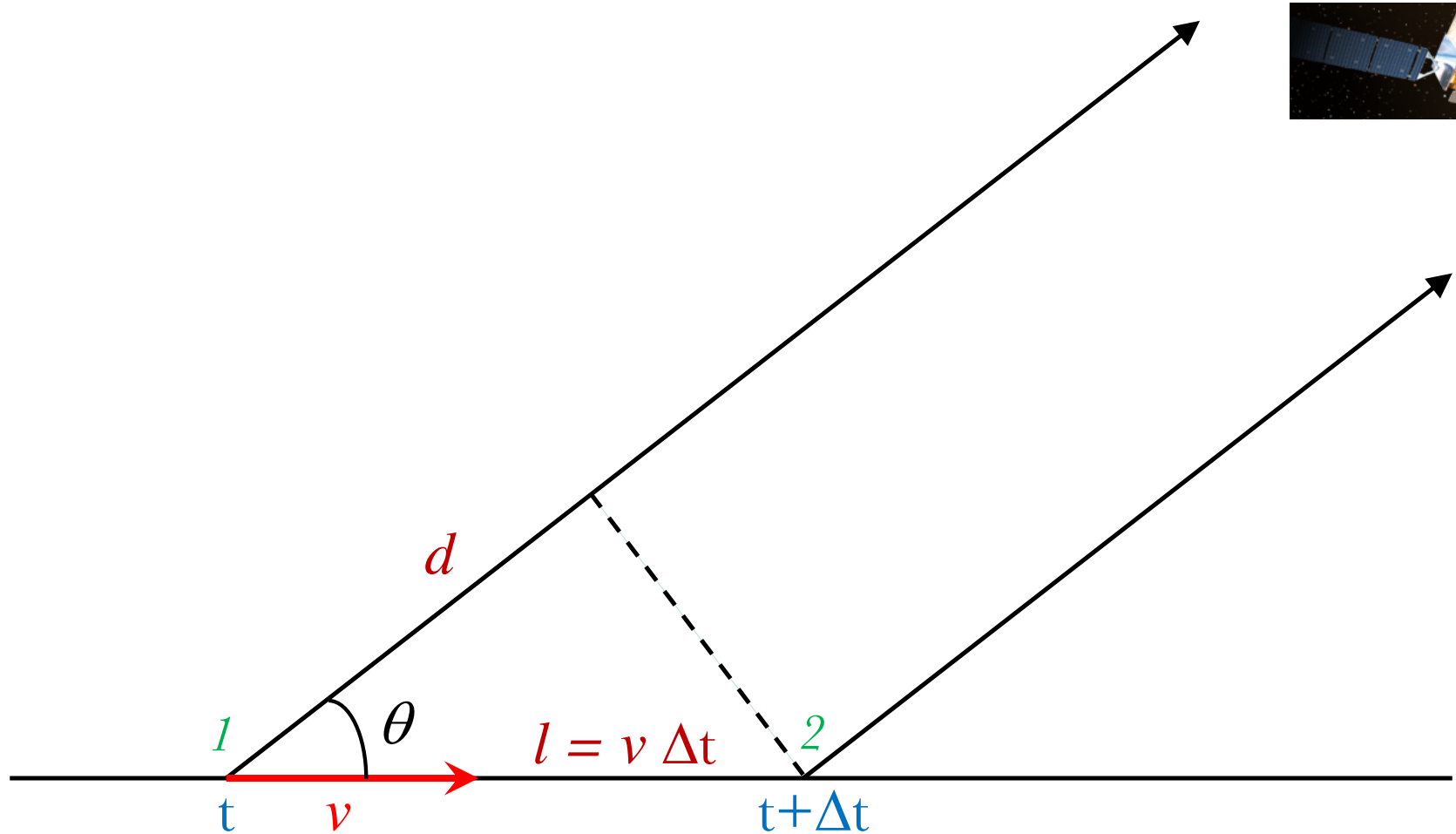
Hence, the observed frequency of ω in the K frame is

$$\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\gamma[1 - (v/c) \cos \theta]} . \quad (17)$$

This is the **relativistic Doppler formula**. It applies to all reference frames and *even in material media* (refractive index does not impact relativity).

- The $1 - (v/c) \cos \theta$ portion appears in non-relativistic formulations, however, the γ factor is a purely relativistic influence. Accordingly, for $\theta = \pi/2$ there is a Doppler reduction in frequency due to relativity; this is called the **transverse Doppler shift**. This serves as a bifurcation angle, separating two key domains. For $0 < \theta \lesssim \pi/2$ viewing perspectives (motion towards an observer), these yield a **blueshift**, while $\pi/2 \lesssim \theta < \pi$ cases (motion away from an observer) produce a **redshift**.

Doppler Effect Geometry



2 Maxwell's Equations in Covariant Form

Consideration of four-vectors, tensors and their Lorentz transformation is left as a reading assignment: consult Sections 4.2 and 4.3 of Rybicki & Lightman. In particular, see Eq. 4.20 therein for the Lorentz transformation matrix.

2.1 Covariant Equations

Without derivation, we will state various forms. The equation of charge conservation uses the **four-current** $j^\mu = (\rho c, \mathbf{j})$, and is

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$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad \Leftrightarrow \quad \partial_\mu j^\mu = 0 \quad , \quad (18)$$

where the **Einstein summation convention** for indices is employed. For a **four-potential** $A^\mu = (\phi, \mathbf{A})$, the Lorenz gauge condition is

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0 \quad \Leftrightarrow \quad \partial_\mu A^\mu = 0 \quad , \quad (19)$$

and with it, we can condense the scalar and vector wave equations for the field into the compact form

$$\nabla^2 A^\beta - \frac{1}{c^2} \frac{\partial^2 A^\beta}{\partial t^2} \equiv -\partial_\mu \partial^\mu A^\beta = -\frac{4\pi}{c} j^\beta \quad . \quad (20)$$

In the absence of charges, this corresponds to light.

Now, we seek a tensor that incorporates the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ themselves. Since they are obtained from derivatives of the scalar and vector potentials, we need to construct something involving derivatives of the four-potential. There should be only six independent elements, corresponding to components of the fields. This suggests the *anti-symmetric tensor*

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \quad (21)$$

A symmetric tensor would have ten independent components. The anti-symmetric choice automatically has zeroes along the diagonal, and the anti-symmetry captures the curl character of the Maxwell field. Evaluation of

this construct gives

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} , \quad F^{\mu\nu} = -F_{\mu\nu} . \quad (22)$$

This proves to be a suitable choice because it yields compact forms for Maxwell's equations. The two inhomogeneous equations (with source terms)

$$\nabla \cdot \mathbf{E} = 4\pi\rho \quad , \quad \nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \mathbf{j} \quad (23)$$

can be written as

$$\partial^\nu F_{\mu\nu} = \frac{4\pi}{c} j_\mu . \quad (24)$$

Taking a further derivative of this again generates charge conservation.

The homogeneous equations require slightly more work to be expressed compactly. This is facilitated by defining a **dual field-strength tensor**

$$\mathcal{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix} . \quad (25)$$

Here $\epsilon_{\mu\nu\alpha\beta}$ is the totally anti-symmetric tensor of rank 4 whose elements are unity for even permutations and -1 for odd permutations of indices, and zero if any two indices are equal (see Jackson p. 550). Then the equations

$$\nabla \cdot \mathbf{B} = 0 \quad , \quad \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (26)$$

can be cast in the form

$$\partial^\nu \mathcal{F}_{\mu\nu} = 0 . \quad (27)$$

Since both the field strength tensor and its dual are of second rank, the Lorentz transformation between any two frames can be written compactly as

$$F'_{\mu\nu} = \Lambda_\mu^\alpha F_{\alpha\beta} \Lambda_\nu^\beta , \quad (28)$$

where the pre-multiplication is by a transpose matrix.

2.2 Transformation of the Electromagnetic Field

From the Lorentz transformation of the electromagnetic tensor $F_{\mu\nu}$ for a pure boost with velocity $\mathbf{v} = c\boldsymbol{\beta}$, the field components transform as

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$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} \quad , \quad \mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B}) \quad , \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \quad , \quad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}) \quad . \end{aligned} \quad (29)$$

Observe the symmetric role played by $\mathbf{E}$ and $-\mathbf{B}$. It is immediately realized that there is *no such thing as a pure electric or magnetic field in all frames*; they are not Lorentz invariants. Accordingly, we adopt the general term of the **electromagnetic field**.

- Yet, we can identify key Lorentz invariants by forming scalars from the electromagnetic tensor $F_{\mu\nu}$. There are two such invariants that are linearly independent and that are quickly obtained:

$$F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2) \quad , \quad \det F = (\mathbf{E} \cdot \mathbf{B})^2 \quad . \quad (30)$$

The first, the square of F , is an invariant because the contraction process, when applied to Lorentz transformation tensors Λ_{α}^{β} , suitably yields unit tensors. The second results because the determinant of the Λ_{α}^{β} tensors is unity, and determinants multiply.

* Both these invariants can be established directly using Eq. (29).

The invariance of $\mathbf{B}^2 - \mathbf{E}^2$ just expresses conservation of “internal” energy: while boosts can enhance the total energy perceived by an observer, the relative content in the magnetic and electric components is fixed.

The invariance of $\mathbf{E} \cdot \mathbf{B}$ connects to the vector identity

$$(\mathbf{E} \cdot \mathbf{B})^2 = E^2 B^2 - (\mathbf{E} \times \mathbf{B}) \cdot (\mathbf{E} \times \mathbf{B}) \quad , \quad (31)$$

and essentially indicates that the zero rest mass of the electromagnetic field is conserved under Lorentz transformations: the first term on the RHS represents energy content, and the second, Poynting flux or momentum content.

* If the electric field dominates (or is inferior to) the magnetic field, it does so in all frames of reference.

* If there are components of the two fields parallel to each other (i.e, $\mathbf{E} \cdot \mathbf{B} \neq 0$), this is also true in all frames of reference.

• Interpreting these invariants is enhanced using *gedanken experiments* involving static charge distributions with zero current, or counterstreaming currents of charges of opposite sign but zero net charge. Both circumstances possess $\mathbf{E} \cdot \mathbf{B} = 0$, which cannot be transformed away by a boost.

* The zero values of both invariants constitutes the statement that light is perceived as light in all frames of reference.



• For a physical understanding of these transformations, consider a capacitance established by two planes of charges separated by distance d in the K frame, and oriented perpendicular to the x direction. The electric field $E = 4\pi\sigma$ is uniform between these planes whose surface charge density is σ . From the perspective of the K' frame, the capacitor moves with speed v along x , so that the planes are now separated by d/γ . Yet, the surface charge density is unchanged $\sigma' = \sigma$, since the areas do not change, and neither does the total charge on each plane. Then

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$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \quad (32)$$

automatically follows.

• Now orient the capacitor planes parallel to the x axis. Since the lengths are contracted, the charge density is increased to $\sigma' = \gamma\sigma$ in the K' frame. Yet, we now also have a **surface current** density $\mu' = -\sigma'v \equiv -\gamma\sigma v$ in the K' frame, which gives rise to a magnetic field $B'_z = -(4\pi/c)\mu'$ in the z -direction. It then follows that

$$\mathbf{E}'_{\perp} = \gamma\mathbf{E}_{\perp} \quad , \quad \mathbf{B}'_{\perp} = -\gamma\boldsymbol{\beta} \times \mathbf{E}_{\perp} \quad . \quad (33)$$

Other constructions combined with superposition of fields can yield the total ensemble of transformation elements.