12. RADIATION FROM RELATIVISTIC CHARGES

Matthew Baring — Lecture Notes for PHYS 532, Spring 2023

1 Radiation from Relativistic Charges

For a final foray, we now consider relativistic charges and the general formalism for their radiation. This means that the dipole approximation must be relinquished. Hereafter we will consider just a single charge, and the obtained results can then be summed for an ensemble of charges.

L&L Sec. 73

1.1 Radiated Power

We seek first the **total power** of radiation from a relativistic charge. The simplest way to proceed is to Lorentz transform quantities from the frame where the charge is instantaneously at rest. In that frame, the Larmor dipole formalism applies if the charge accelerates, and one can write down

$$\frac{d\mathcal{E}_{\rm rad}}{dt} \;=\; \frac{2q^2}{3c^3}\, a^2 \quad , \quad \frac{d\mathbf{P}_{\rm rad}}{dt} \;=\; \mathbf{0} \quad . \tag{1}$$

Here $q\mathbf{a} = \ddot{\mathbf{d}}$ defines the dipole moment acceleration vector. The second relation is for the total radiated momentum, and is an obvious result given the symmetry of the dipole antenna pattern. These can be combined and expressed in covariant form, noting that $dx^\alpha = u^\alpha ds$:

$$dP^\alpha_{\rm rad} \;=\; -\frac{2q^2}{3}\, a^\beta a_\beta \, dx^\alpha \quad \text{for} \quad a^\beta a_\beta \;=\; -\frac{a^2}{c^4} \quad \text{with} \quad a^\beta \;=\; \frac{du^\beta}{ds} \quad . \tag{2}$$

Remember that in the frame of instantaneous rest, the four-acceleration can be written $a^\beta = (0, a/c^2, 0, 0)$ and so the scalar product for the acceleration is both obvious, and also a Lorentz invariant. It is used to substitute for the magnitude a of the acceleration vector. Thus, the verity of the $\alpha = 0$ form of Eq. (2) is apparent. Since $dx^i = 0$ in the instantaneous rest frame, the space components are trivially zero and the momentum equation is also verified. The result is a fully covariant form for radiated 4-momentum.[1]

This result is clearly applicable in any inertial frame, and so can be Lorentz-transformed to frames where the charge is not at rest. To render this more useful, the 4-acceleration can now be replaced by the covariant Lorentz force:

$$a^\beta \;\equiv\; \frac{du^\beta}{ds} \;=\; \frac{q}{mc^2}\, F^{\beta\mu} u_\mu \quad \Rightarrow \quad dP^\alpha_{\rm rad} \;=\; -\frac{2q^4}{3m^2c^4}\,\big(F^{\beta\mu}u_\mu\big)\big(F_{\beta\nu}u^\nu\big)\, dx^\alpha \;. \tag{3}$$

The electromagnetic field tensor has thus appeared, and so the radiated energy and momentum can be directly connected to charge motions in electric and magnetic fields, and in any frame of reference. Setting $\boldsymbol{\beta} = \mathbf{v}/c$,

$$F^{\beta\mu} u_\mu \;=\; \gamma\Big(\mathbf{E}\cdot\boldsymbol{\beta},\, \mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}\Big) \quad . \tag{4}$$

This can be substituted into the radiation equation and the $\alpha = 0$ energy component formed, yielding the **radiated power**:

$$\frac{d\mathcal{E}_{\rm rad}}{dt} \;=\; \frac{2q^4}{3m^2c^3}\,\gamma^2 \bigg\{ \Big(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}\Big)^2 - \big(\mathbf{E}\cdot\boldsymbol{\beta}\big)^2 \bigg\} \quad . \tag{5}$$

When $\beta \to 0$ and $\gamma \to 1$, this reduces to the original rest form using the correspondence $q\mathbf{E}/m \to \mathbf{a}$ for acceleration induced by an electric field.

- The power (integrated over all solid angles) thus generally scales as γ^2. This is to be expected, with contributions of Lorentz factors from the Doppler shift and the time dilation. However, if $\mathbf{B} = \mathbf{0}$, and the motion is exactly along a uniform electric field vector ($\mathbf{v} \parallel \mathbf{E}$), then an additional $1/\gamma^2$ factor appears so that the radiated power is an invariant during the motion.

[1] L&L have an error of a factor of $1/c$ in this and a subsequent equation.

1.2 Angular Emission Profiles

We now turn our attention to the angular distribution of the radiation. This cannot easily be worked in terms of covariant forms, so we revert to the Liénard-Wiechart potentials. Since we are interested in far-zone results, one retains terms of leading order $1/R$, so that

$$\mathbf{A}_{\rm rad} \; = \; \frac{q\,\boldsymbol{\beta}}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})} \quad \Rightarrow \quad \mathbf{E}_{\rm rad} \; = \; \frac{q}{cR}\,\frac{\mathbf{n}\times\big\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\big\}}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3} \quad , \tag{6}$$

both being evaluated for retarded times $t' = t - R/c$. The magnetic field is just $\mathbf{B}_{\rm rad} = \mathbf{n}\times\mathbf{E}$. Observe that the Larmor dipole formula can be recovered when $\beta \ll 1$ using $q\dot{\boldsymbol{\beta}}c = \ddot{\mathbf{d}}$. The energy radiated into solid angle $d\Omega$ is simply $d\mathcal{E}_{\rm rad} = (c/4\pi)E^2R^2\,d\Omega dt$. Performing the vector algebra,

$$\frac{d\mathcal{E}_{\rm rad}}{d\Omega dt} \; = \; \frac{q^2}{4\pi c}\left\{ \frac{2(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\,(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5} + \frac{(\dot{\boldsymbol{\beta}})^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^4} - \frac{(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})^2}{\gamma^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^6} \right\}_{\rm ret} \quad . \tag{7}$$

The first term is small in the non-relativistic limit, wherein the remaining two give a result proportional to $|\dot{\boldsymbol{\beta}}\times\mathbf{n}|^2$, as expected.

- The appearance of the multiple factors $1-\mathbf{n}\cdot\boldsymbol{\beta}$ in the denominators of the terms defines a strong enhancement of the radiation in directions close to the charge's instantaneous $\boldsymbol{\beta}$. This is due to a combination of time dilation, Doppler shifting, and light aberration influences. When $\beta \sim 1$, the emission is highly-collimated because $1-\beta\cos\theta \approx (1/\gamma^2+\theta^2)/2$, with an angle

$$\theta \; \sim \; \frac{1}{\gamma} \; = \; \sqrt{1-\beta^2} \tag{8}$$

that defines the opening half-angle of the **Lorentz cone** of the radiation.

- There are clearly two directions where the emission is identically zero, those where the vector $\mathbf{n}-\boldsymbol{\beta}$ is parallel or anti-parallel to the acceleration vector, i.e. $\dot{\boldsymbol{\beta}}$. This defines relativistic aberration of the antenna pattern.

To determine the total emission, integrated over time, it is appropriate to work with the t' retarded time variable. Thus Eq. (7) can be used to evaluate

$$\frac{d\mathcal{E}_{\rm rad}}{d\Omega} \; = \; \int \frac{d\mathcal{E}_{\rm rad}}{d\Omega dt}\,(1-\mathbf{n}\cdot\boldsymbol{\beta})\,dt' \quad \text{as} \quad dt \; = \; (1-\mathbf{n}\cdot\boldsymbol{\beta})\,dt' \quad . \tag{9}$$

In general, this has to be computed numerically.

• To illustrate the character of these angular distributions, we highlight two special cases. The first is when the *charge's velocity and acceleration are parallel*. Then the magnetic field of the wave is

$$\mathbf{B}_{\rm rad} \;=\; \frac{q}{cR}\,\frac{\dot{\boldsymbol{\beta}}\times\mathbf{n}}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3} \;=\; \frac{q\dot{\beta}}{cR}\,\frac{\hat{\boldsymbol{\beta}}\times\mathbf{n}}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\quad . \tag{10}$$

If θ is the polar angle relative to $\mathbf{v}$ (i.e., $\cos\theta = \mathbf{n}\cdot\hat{\boldsymbol{\beta}}$, contrasting the dipole formalism where Θ is the angle between $\mathbf{n}$ and $\dot{\boldsymbol{\beta}}$), then in this case $\theta = \Theta$ and the radiation angular profile is (writing $\dot{\beta} \to \dot{\beta}_{\parallel}$)

$$\frac{d\mathcal{E}_{\rm rad}}{d\Omega dt} \;=\; \frac{q^2}{4\pi c}\,\frac{(\dot{\beta}_{\parallel})^2\,\sin^2\theta}{(1-\beta\cos\theta)^6}\quad . \tag{11}$$

This is depicted in the following Figure. It is applicable to a **linear acceleration** of a charge in a uniform $\mathbf{E}$ field, and a 1D relativistic SHO. The angle-integrated power $d\mathcal{E}_{\rm rad}/dt$ is routinely obtained in analytic form.

Plot: Radiation from a Linearly-Accelerated Charge

• In contrast, if *the acceleration is perpendicular to the charge's velocity*, as is the circumstance for **synchrotron radiation**, then the aberration of the antenna pattern is given by (writing $\dot{\beta} \to \dot{\beta}_{\perp}$)

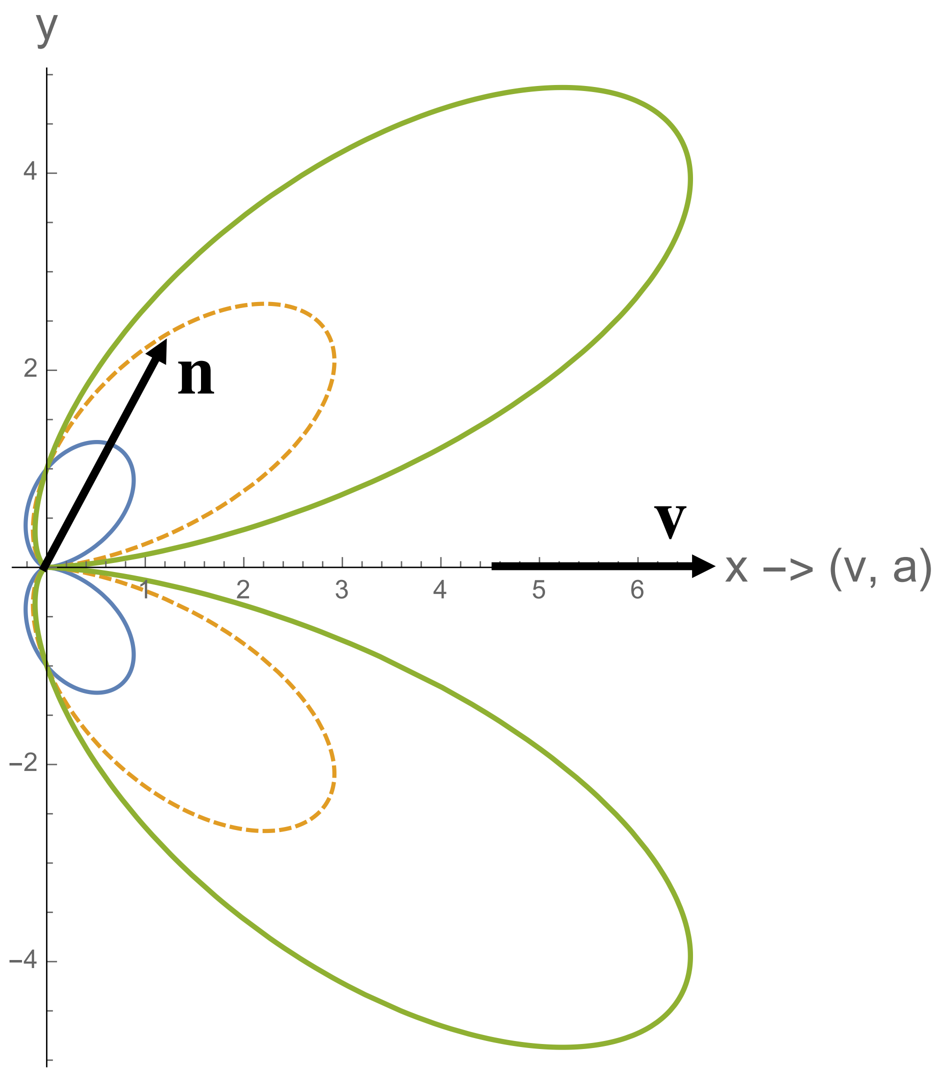
$$\frac{d\mathcal{E}_{\rm rad}}{d\Omega dt} \;=\; \frac{q^2}{4\pi c}\,(\dot{\beta}_{\perp})^2\left\{\frac{1}{(1-\beta\cos\theta)^4} - \frac{\sin^2\theta\,\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^6}\right\}\quad . \tag{12}$$

Here ϕ is the azimuthal angle of $\mathbf{n}$ relative to the plane defined by $\mathbf{v}$ and $\mathbf{a}$, i.e., about the $\mathbf{v}\times\mathbf{a}$ vector. For the case of $\phi = 0$, this is illustrated in the corresponding Figure.

Plot: Radiation: charge accelerating orthogonal to its velocity

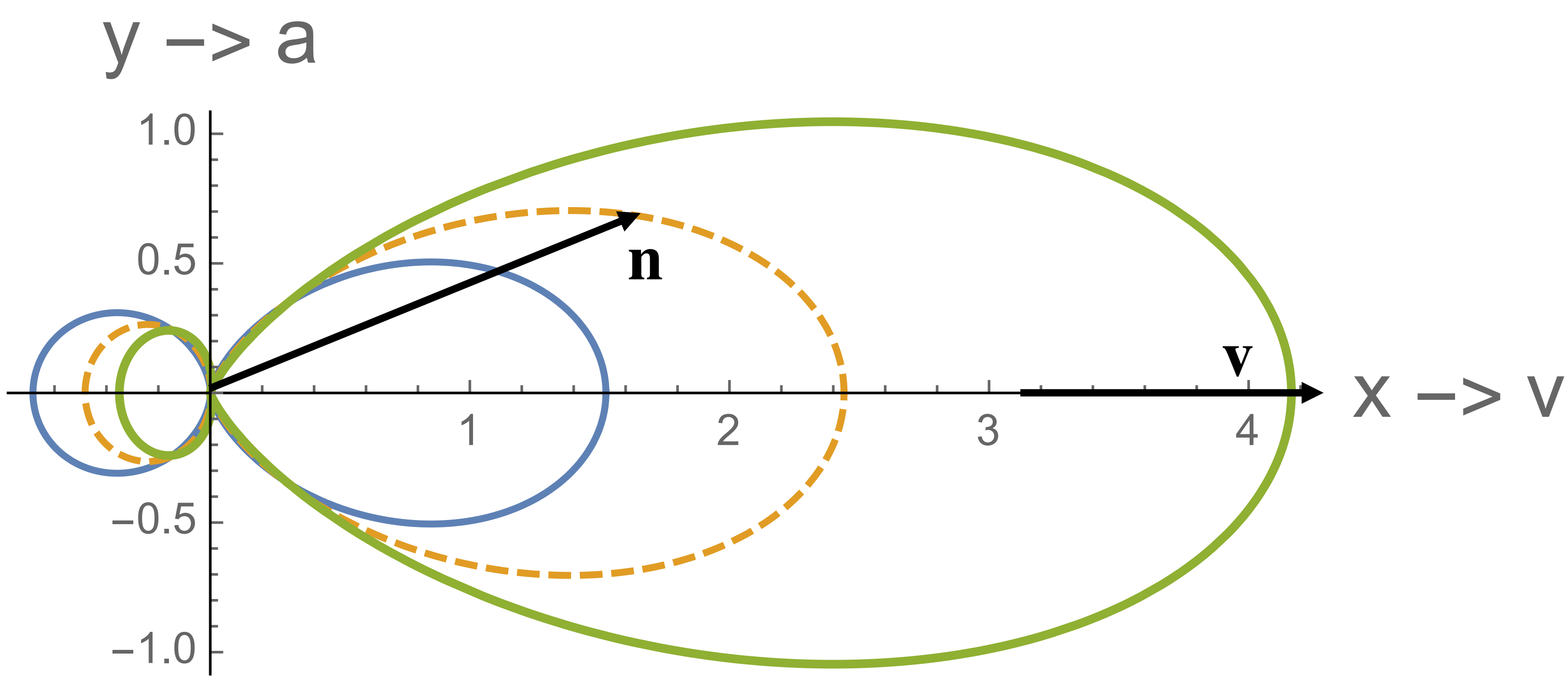
In this case, as for the first, the distortion of the antenna pattern lobes in the direction of $\boldsymbol{\beta}$ is evident as the charge becomes relativistic, and the zeroes are when $\dot{\boldsymbol{\beta}} \propto \mathbf{n} - \boldsymbol{\beta}$.

Radiation from a Linearly-Accelerated Charge



- The radiation pattern (section) for a charge accelerating parallel to its velocity **v** for three speeds:
  - $\beta = 0.2$ (small blue lobes),
  - $\beta = 0.4$ (dashed orange)
  - $\beta = 0.5$ (large green).

Radiation: Acceleration Perpendicular to Velocity



- The radiation pattern (section) for a charge accelerating perpendicular to its velocity **v** for three speeds:
  - $\beta = 0.1$ (small blue lobes),
  - $\beta = 0.2$ (dashed orange)
  - $\beta = 0.3$ (large green).

2 Synchrotron Radiation

As a final example, we consider **synchrotron radiation**, the relativistic analog of the cyclotron mechanism. The motion is described by the Lorentz force that for the special case of circular motion provides an acceleration orthogonal to the velocity at any time:

$$\mathbf{a} \;=\; \mathbf{v}\times\boldsymbol{\omega}_{\rm B} \quad , \quad \omega \;\equiv\; \left|\boldsymbol{\omega}_{\rm B}\right| \;=\; \frac{qB}{\gamma mc} \quad . \tag{13}$$

The gyroradius is pc/qB and the total radiated power can be quickly determined from Eq. (5):

$$\frac{d\mathcal{E}_{\rm rad}}{dt} \;=\; \frac{2q^4}{3m^2c^3}\,\gamma^2\big(\boldsymbol{\beta}\times\mathbf{B}\big)^2 \;\to\; \frac{2r_q^2c}{3}\,\gamma^2\beta^2\,B^2 \quad . \tag{14}$$

Obtaining the angular distribution of the synchrotron radiation is slightly more involved. The instantaneous distribution is given in Eq. (12). Yet, as the charge moves in a circle, the definition of angles changes. Physically, the circular motion is not resolved, so it is motivated to integrate over the gyroperiod $2\pi/\omega_{\rm B}$, i.e. azimuths $\phi_g = \omega_{\rm B}t$. Let the observer direction $\mathbf{n}$ be inclined at an angle θ to the plane of the circular orbit, and the projection of $\mathbf{n}$ onto the orbit $(y,\, z)$ plane correspond to azimuth $\phi_g = 0$. Then the instantaneous direction between $\mathbf{n}$ and $\mathbf{v}$ satisfies $\boldsymbol{\beta}\cdot\mathbf{n} = \beta\cos\theta\cos\phi_g$ as

$$\hat{\mathbf{v}} \;=\; \cos\phi_g\,\hat{y} + \sin\phi_g\,\hat{z} \quad \text{and} \quad \hat{\dot{\mathbf{v}}} \;=\; -\sin\phi_g\,\hat{y} + \cos\phi_g\,\hat{z} \quad , \tag{15}$$

and $\mathbf{n} \;=\; \cos\theta\,\hat{y} + \sin\theta\,\hat{x}$.

Plot: Synchrotron radiation geometry

It follows that the time-averaged value of the intensity in Eq. (7) is

$$\left\langle\frac{d\mathcal{E}_{\rm rad}}{d\Omega dt}\right\rangle_t \;=\; \frac{r_q^2c}{8\pi^2}\,\frac{\beta^2}{\gamma^2}\,B^2\int_0^{2\pi}\frac{(1-\beta^2)\sin^2\theta + (\beta-\cos\theta\cos\phi_g)^2}{(1-\beta\cos\theta\cos\phi_g)^5}d\phi_g \quad . \tag{16}$$

The integral can be evaluated analytically.

$$\left\langle\frac{d\mathcal{E}_{\rm rad}}{d\Omega dt}\right\rangle_t \;=\; \frac{r_q^2c}{32\pi}\,\frac{\beta^2}{\gamma^2}\,B^2\left\{\frac{8-4\cos^2\theta-\beta^2(1+3\beta^2)\cos^4\theta}{(1-\beta^2\cos^2\theta)^{7/2}}\right\} \quad . \tag{17}$$