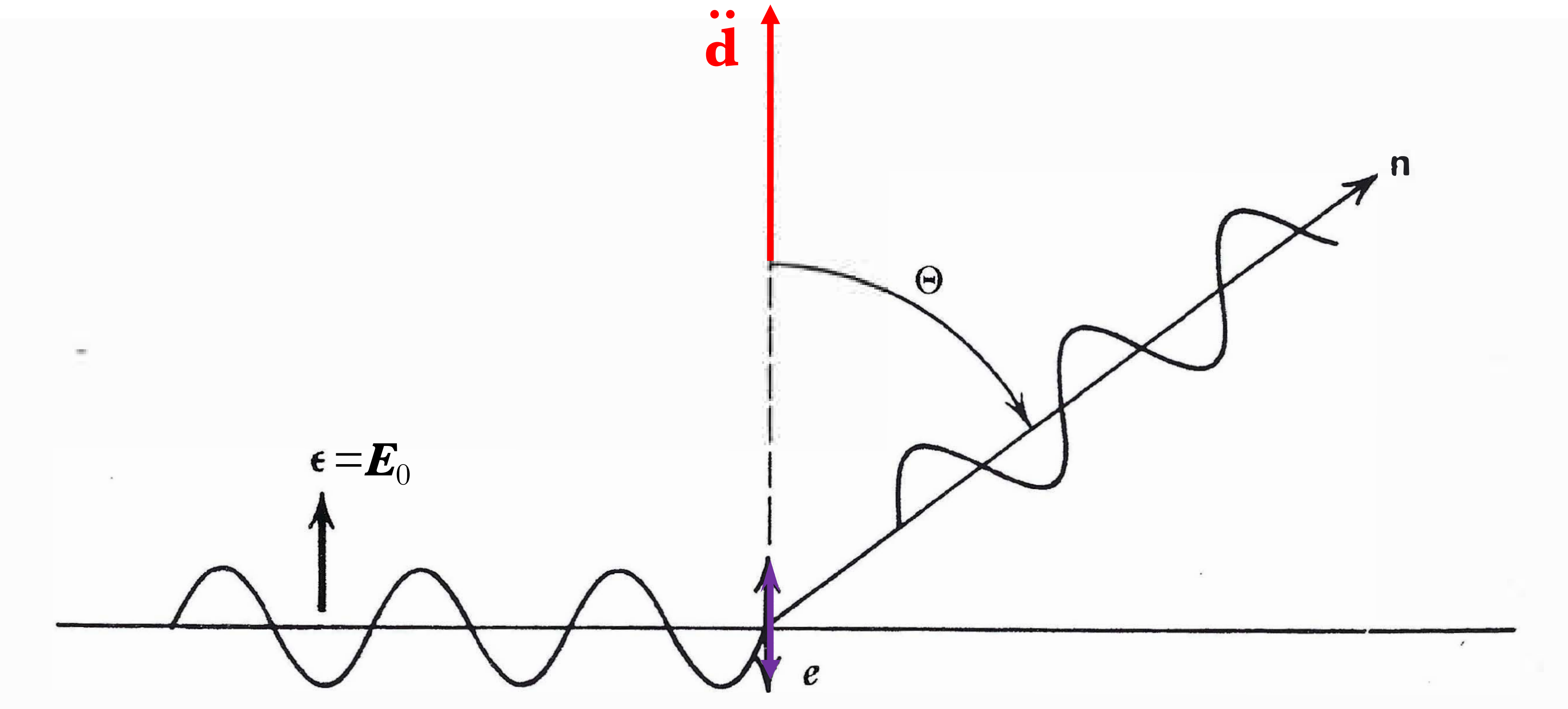
Thomson Scattering Geometry



- Scattering of linearly-polarized radiation by an electron.
- Fig. 3.6 of Rybicki & Lightman: *Radiative Processes in Astrophysics* (1979).

• The effective coupling between the incoming and outgoing radiation is captured in the **cross section** σ. This measures the ratio between the radiated intensity and the incoming intensity, the latter being benchmarked by the magnitude $S = |\mathbf{S}| = c|\mathbf{E}|^2/4\pi$ of the Poynting flux vector. We therefore define the **differential cross section** $d\sigma/d\Omega$ by

$$\left\langle \frac{d\mathcal{E}_{\rm rad}}{d\Omega\, dt} \right\rangle_t = \big\langle |\mathbf{S}| \big\rangle_t \frac{d\sigma}{d\Omega} \quad , \tag{23}$$

where time averages over the electromagnetic wave properties are taken. Since the Poynting flux is energy per unit area per unit time, the cross section is an *effective area*. We seek to determine $d\sigma/d\Omega$.

Given the time variation of the dipole moment, the Larmor formalism quickly yields the radiated intensity:

$$\frac{d\mathcal{E}_{\rm rad}}{d\Omega\, dt} = \frac{|\ddot{\mathbf{d}} \times \mathbf{n}|^2}{4\pi c^3} = \frac{q^4}{4\pi m^2 c^3} \big|\mathbf{E} \times \mathbf{n}\big|^2 \quad . \tag{24}$$

As usual, $\mathbf{n}$ is the unit vector in the direction of the outgoing wave, which makes an angle Θ to polarization direction $\mathbf{E}_0$. The time average over a wave period $2\pi/\omega$ can be taken:

$$\left\langle \frac{d\mathcal{E}_{\rm rad}}{d\Omega\, dt} \right\rangle_t = \frac{q^4}{m^2 c^4} \underbrace{\left\{ \frac{c}{4\pi} |\mathbf{E}_0|^2 \left\langle \cos^2(\omega t + \alpha) \right\rangle_t \right\}}_{\langle |\mathbf{S}| \rangle_t} \sin^2\Theta \quad . \tag{25}$$

The quantity inside the curly braces is the wave's time-averaged Poynting flux $\langle |\mathbf{S}| \rangle_t$, and so using Eq. (23) we infer the differential cross section:

$$\frac{d\sigma}{d\Omega} = r_q^2 \sin^2\Theta \quad , \quad r_q = \frac{q^2}{mc^2} \quad . \tag{26}$$

As for cyclotron radiation, r_q is the classical electrodynamic radius, and for the case of the electron it is $r_0 = e^2/m_e c^2 = 2.818 \times 10^{-13}\,$cm. Integration over solid angles yields the **Thomson cross section**

$$\sigma = \frac{8\pi r_q^2}{3} = 6.652 \times 10^{-24}\,\mathrm{cm}^2 \text{ for electrons} \quad . \tag{27}$$

Again it applies to a purely linearly-polarized wave, and the light deflection mechanism is referred to as **Thomson scattering**.

• The scattering of the wave preserves the frequency of the wave (obvious from the power spectrum considerations), the electric field polarization, and the electric field amplitude, to leading order. The latter point is not obvious, but is expected if the charge only performs small amplitude oscillations.

• Finally consider the scattering of unpolarized light. Here the orientations of the $\mathbf{E}_0$ vector are uniformly distributed in the (y, z)-plane. Let θ be the angle between directions of the incident and scattered waves. Then for azimuthal angle ϕ of the incoming wave $\mathbf{E} = E_0 \hat{e}$ vector about the x-axis,

$$\begin{aligned} \hat{e} &= \cos\phi\,\hat{y} + \sin\phi\,\hat{z} \quad , \\ \mathbf{n} &= \cos\theta\,\hat{x} + \sin\theta\left(\cos\phi_n\,\hat{y} + \sin\phi_n\,\hat{z}\right) \quad . \end{aligned} \tag{28}$$

Here ϕ_n is the azimuthal angle of the final wave.

Plot: Unpolarized radiation scattering geometry.

The time average is as before, and the azimuth average of the angle factor $|\ddot{\mathbf{d}} \times \mathbf{n}|^2 \,/\, |\ddot{\mathbf{d}}|^2$ in the Thomson differential cross section can then be formed:

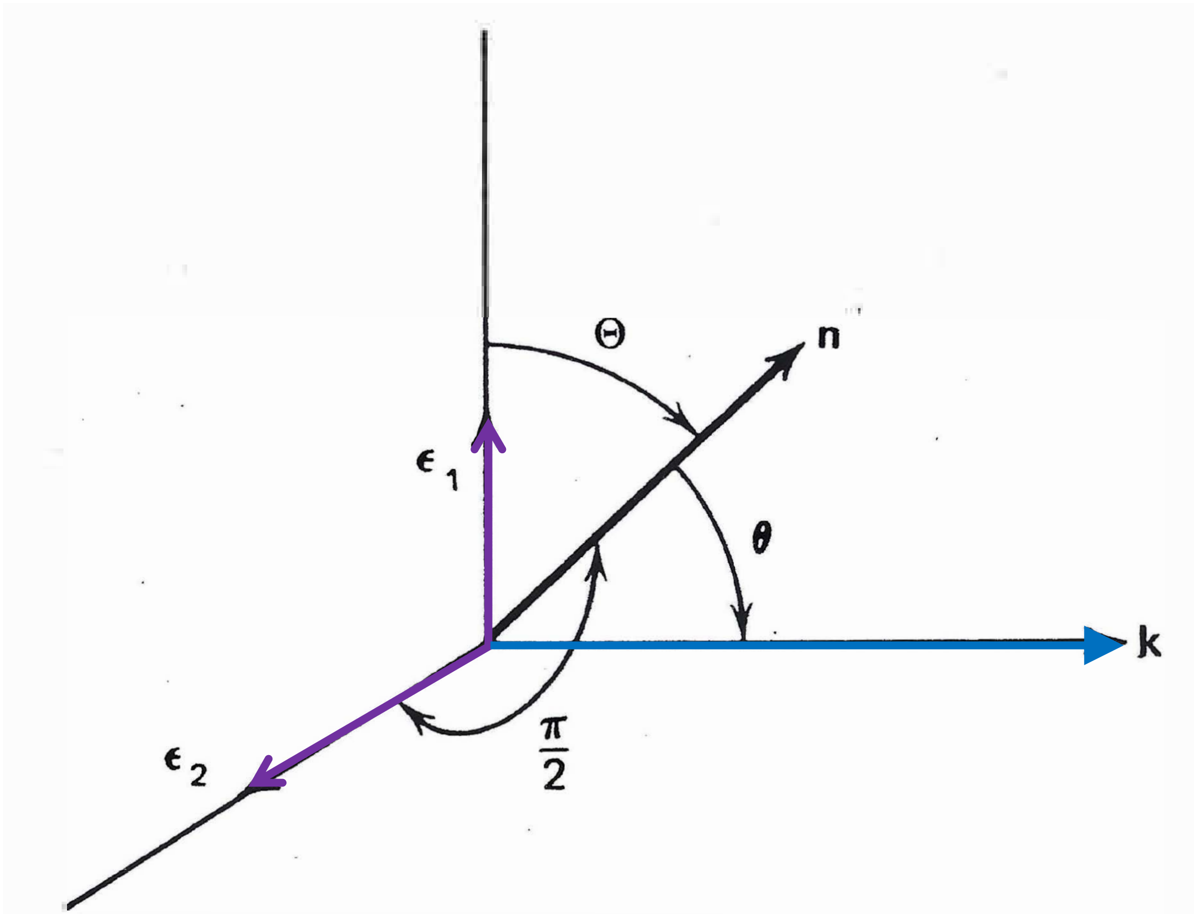
$$\begin{aligned} \left\langle \sin^2\Theta \right\rangle_\phi &= 1 - \left\langle (\hat{e}\cdot\mathbf{n})^2 \right\rangle_\phi \\ &= 1 - \sin^2\theta \left(\left\langle \cos^2\phi \right\rangle_\phi \cos^2\phi_n + \left\langle \sin^2\phi \right\rangle_\phi \sin^2\phi_n \right) \\ &= 1 - \frac{1}{2}\sin^2\theta \;=\; \frac{1}{2}\left(1 + \cos^2\theta\right) \quad . \end{aligned} \tag{29}$$

Note that the $\hat{y}\cdot\hat{z}$ cross term is identically zero when averaging over ϕ. Therefore the classical differential cross section for unpolarized light is

$$\frac{d\sigma}{d\Omega} = \frac{r_q^2}{2}\left(1 + \cos^2\theta\right) \quad , \quad r_q = \frac{q^2}{mc^2} \quad . \tag{30}$$

There is now no zero in radiated intensity, because the antenna pattern (*like a doughnut or toroid*) has been rotated about the x-axis. Integrating over all angles, reproduces the total Thomson cross section, as it should.

Unpolarized Thomson Scattering



- Geometry of unpolarized Thomson scattering by an electron: ε_1 and ε_2 are the two linear polarizations of the incoming waves.
- Fig. 3.7 of Rybicki & Lightman: *Radiative Processes in Astrophysics* (1979).

3 Bremsstrahlung

Motion in the Coulomb field implicitly involves acceleration of charges in unbound hyperbolic trajectories, and so naturally produces electromagnetic radiation. Since the ballistic charge is the one that emits, if it has a massive target charge, then it must slow down somewhat. Therefore the process is termed **bremsstrahlung**, or "braking radiation."

Plot: Bremsstrahlung geometry

• The acceleration is greatest at **perielektron**, so the epoch of nearest passage dominates the radiation signal, its duration and thus its characteristic emission frequency. The charge's trajectory is planar for a single target charge, and so the emitted radiation is 100% linearly-polarized in this plane.

3.1 Bremsstrahlung in the Dipole Domain

In general, the case of greatest interest is when a beam of particles is projected toward a target central charge Q, rather than an isolated single collision between two charges. Therefore, we explore ensemble-averaged interactions, with the beam represented by a cylindrical morphology. The beam will be presumed to extend to $b_{\rm max}$ in **impact parameter** b, where this scale may represent a mean separation of the ballistic charges in a plasma. It is also azimuthally symmetric. The non-relativistic single-particle emission rate is **L&L Sec. 68**

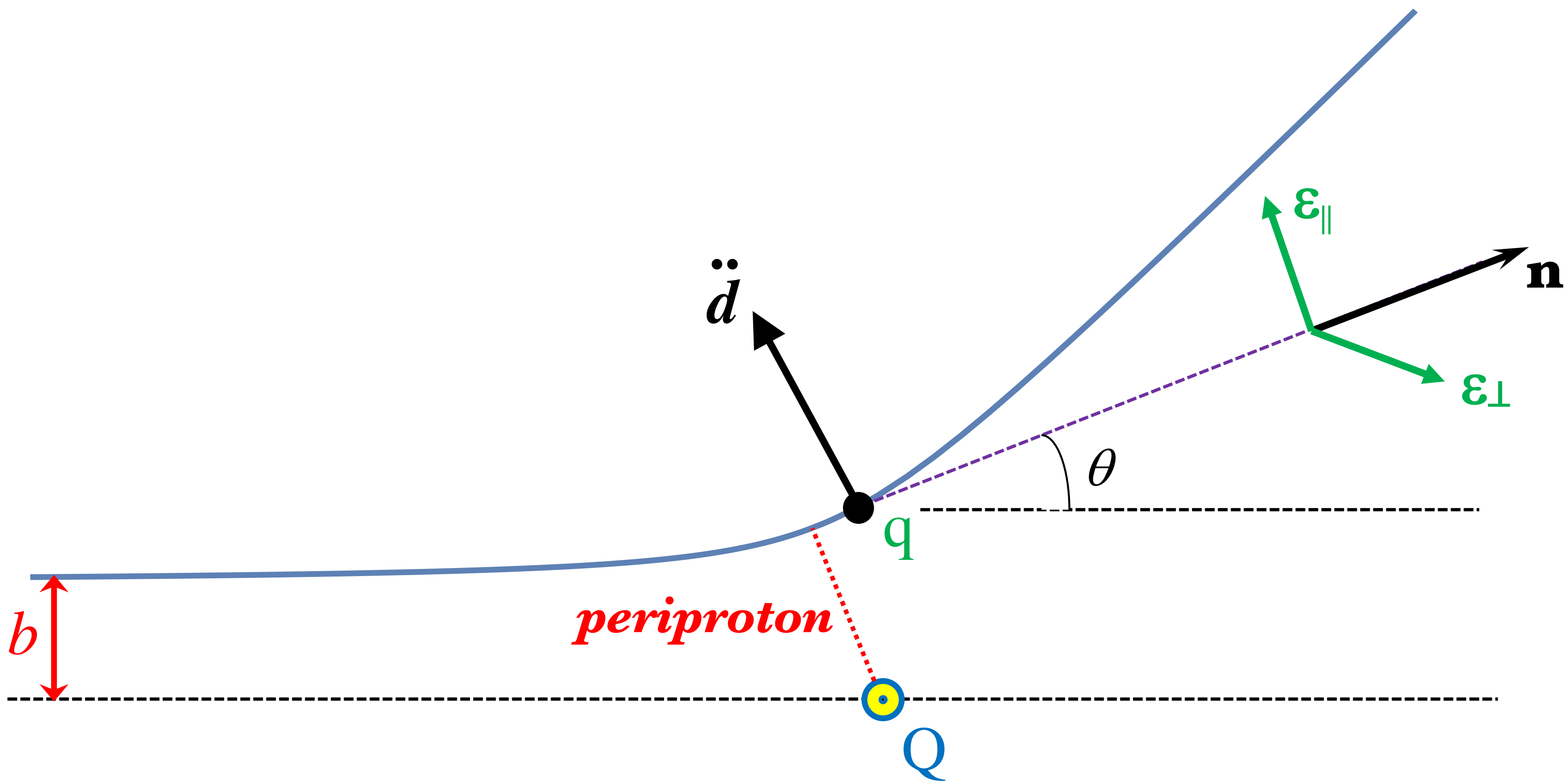
$$\frac{d\mathcal{E}_{\rm rad}}{d\Omega\, dt} \;=\; \frac{|\ddot{\mathbf{d}}(t)\times\mathbf{n}|^2}{4\pi c^3} \quad . \tag{31}$$

The time dependence $\mathbf{d}(t)$ is incurred as a ballistic charge follows its curved path. One is interested in the total emission, and so we integrate over the complete duration of its path. The azimuthal average is also taken:

$$\begin{aligned} \frac{d\mathcal{E}_{\rm rad}}{d\Omega} &= \int_0^{b_{\rm max}} \frac{2\pi b\, db}{\mathcal{A}} \int_{-\infty}^{\infty} \Bigl\langle \frac{d\mathcal{E}_{\rm rad}}{d\Omega\, dt} \Bigr\rangle_\phi dt \\ &= \frac{1}{2c^3\mathcal{A}} \int_0^{b_{\rm max}} b\, db \int_{-\infty}^{\infty} \bigl\langle |\ddot{\mathbf{d}}(t)\times\mathbf{n}|^2 \bigr\rangle_\phi\, dt \quad . \end{aligned} \tag{32}$$

Thus $\langle \dots \rangle_\phi$ represents the azimuthal integration, and $\mathcal{A} = \pi b_{\rm max}^2$ is the normalizing cross-sectional area of the beam.

Bremsstrahlung Geometry



- Bremsstrahlung dipole emission geometry for light in **n** direction and two possible polarizations ε_i . This is for a charge q deflecting off charge Q via the Coulomb interaction, with impact parameter b.

The azimuthal averages can be expressed in terms of two dipole moment quantities and the *observer angle* θ to the beam. Let the incoming beam be parallel to the x-direction. The cross product is

$$\left|\ddot{\mathbf{d}}(t) \times \mathbf{n}\right|^2 \;=\; \left|\ddot{\mathbf{d}}(t)\right|^2 - \left|\mathbf{n} \cdot \ddot{\mathbf{d}}(t)\right|^2 \quad . \tag{33}$$

The dipole moment has components in all three coordinate directions. The squares sustain pure quadratic terms, but the cross terms average to zero when sampling azimuthal angles uniformly, since

$$\ddot{\mathbf{d}} \;=\; \left(\ddot{d}_x,\, \ddot{d}_\perp \cos\phi,\, \ddot{d}_\perp \sin\phi\right) \quad . \tag{34}$$

Hence

$$\left\langle \ddot{d}_x \ddot{d}_y \right\rangle_\phi \;=\; \left\langle \ddot{d}_y \ddot{d}_z \right\rangle_\phi \;=\; \left\langle \ddot{d}_z \ddot{d}_x \right\rangle_\phi \;=\; 0 \quad . \tag{35}$$

Azimuthal symmetry renders the y and z squares identical, and so only the second derivatives of the total dipole moment and the x-component are independent quantities at any time. Then

$$\left\langle (\ddot{d}_y)^2 \right\rangle_\phi \;=\; \left\langle (\ddot{d}_z)^2 \right\rangle_\phi \;=\; \frac{1}{2}\left\{ (\ddot{d})^2 - (\ddot{d}_x)^2 \right\} \;\equiv\; \frac{1}{2}\,(\ddot{d}_\perp)^2 \quad . \tag{36}$$

Observe that both $(\ddot{d}_x)^2$ and $(\ddot{d})^2$ are independent of the azimuth, and just depend on the time variable. The observer direction vector is

$$\mathbf{n} \;=\; \left(\cos\theta,\, \sin\theta\cos\phi_o,\, \sin\theta\sin\phi_o\right) \quad . \tag{37}$$

In all generality, one can choose $\phi_o = 0$. The averages of Eq. (33) distill into

$$\left\langle |\ddot{\mathbf{d}}(t) \times \mathbf{n}|^2 \right\rangle_\phi \;=\; \frac{1}{2}\left\{ (\ddot{d})^2 + (\ddot{d}_x)^2 \right\} + \frac{1}{2}\left\{ (\ddot{d})^2 - 3(\ddot{d}_x)^2 \right\} \cos^2\theta \quad . \tag{38}$$

Inserting this into Eq. (32), we arrive at

$$\frac{d\mathcal{E}_{\rm rad}}{d\Omega} \;=\; \alpha_0 + \alpha_2 \underbrace{\frac{3\cos^2\theta - 1}{2}}_{P_2(\cos\theta)} \tag{39}$$

with Legendre functions $P_0(\cos\theta) = 1$ and $P_2(\cos\theta)$ being readily identifiable. The α_i coefficients are

$$\begin{aligned} \alpha_0 \;&=\; \frac{1}{3c^3\mathcal{A}} \int_0^{b_{\max}} b\, db \int_{-\infty}^{\infty} (\ddot{d})^2 \, dt \\ \alpha_2 \;&=\; \frac{1}{6c^3\mathcal{A}} \int_0^{b_{\max}} b\, db \int_{-\infty}^{\infty} \left\{ (\ddot{d})^2 - 3(\ddot{d}_x)^2 \right\} dt \end{aligned} \tag{40}$$

Note that when integrating over emission solid angles $d\Omega = 2\pi d(\cos\theta)$, the term multiplying α_2 is zero due to the orthogonality relation for Legendre polynomials, i.e. with $P_0(\cos\theta) = 1$. Hence, the total bremsstrahlung emission in the dipole approximation is

$$\mathcal{E}_{\rm rad} \;=\; 4\pi\alpha_0 \;=\; \frac{4\pi}{3c^3\mathcal{A}}\int_0^{b_{\rm max}} b\,db \int_{-\infty}^{\infty} \bigl[\ddot{d}(t)\bigr]^2\,dt \quad . \tag{41}$$

The function $\ddot{d}(t)$ depends on the impact parameter b, and can be obtained from the unbound forms for non-relativistic Coulomb trajectories.

- Observe that the angular distribution is symmetric in θ about $\pi/2$. This applies in the non-relativistic dipole approximation because the *forward and backward portions of the radiation antenna signal contribute equally and symmetrically*. This is <u>no longer the case for relativistic bremsstrahlung</u> due to the inherent asymmetry associated with Doppler boosting.

- **Now consider radiation E fields instead**: To develop the **polarization** of the radiation, we will specify two orthogonal polarization states: one in the $\hat{x}-\mathbf{n}$ plane (which we choose to be the $(x,\,y)$ plane) containing the beam and the observer direction (denoted $\|$), and the other orthogonal to this plane (denoted $\perp$). The **E** field vector of the bremsstrahlung satisfies

$$c^2R\,\mathbf{E} \;=\; c^2R\,\bigl(\mathbf{B}\times\mathbf{n}\bigr) \;=\; \bigl(\ddot{\mathbf{d}}\times\mathbf{n}\bigr)\times\mathbf{n} \;=\; \ddot{\mathbf{d}} - \mathbf{n}\,\bigl(\mathbf{n}\cdot\ddot{\mathbf{d}}\bigr) \quad . \tag{42}$$

Squaring this gives

$$\bigl|\mathbf{E}\bigr|^2 \;\propto\; \Bigl|\bigl(\ddot{\mathbf{d}}\times\mathbf{n}\bigr)\times\mathbf{n}\Bigr|^2 \;=\; \bigl|\ddot{\mathbf{d}}(t)\bigr|^2 - \bigl|\mathbf{n}\cdot\ddot{\mathbf{d}}(t)\bigr|^2 \quad , \tag{43}$$

just as it should since the electric and magnetic fields have the same intensity. The angle averaging algebra is just as before, and clearly only the $\bigl\langle(\ddot{d}_z)^2\bigr\rangle_\phi$ contribution gives net polarization perpendicular to the $(x,\,y)$-plane. This is specified in Eq. (36), and so we can recast the result in Eq. (38) as

$$\Bigl\langle\bigl|\bigl(\ddot{\mathbf{d}}\times\mathbf{n}\bigr)\times\mathbf{n}\bigr|^2\Bigr\rangle_\phi \;=\; \underbrace{\frac{1}{2}\Bigl\{(\ddot{d})^2 - (\ddot{d}_x)^2\Bigr\}}_{\mathcal{E}_{\rm rad}^{\perp}} + \underbrace{(\ddot{d}_x)^2 + \frac{1}{2}\Bigl\{(\ddot{d})^2 - 3(\ddot{d}_x)^2\Bigr\}\,\cos^2\theta}_{\mathcal{E}_{\rm rad}^{\|}} \;. \tag{44}$$

Thus, we have identified the algebraic structure associated with the polarization decomposition of the bremsstrahlung emission that now replaces the α_i construction. Therefore, the perpendicular emission state has

$$\frac{d\mathcal{E}_{\text{rad}}^{\perp}}{d\Omega} \;=\; \frac{1}{4c^3}\int_0^{b_{\max}} b\,db \int_{-\infty}^{\infty} \Bigl\{ (\ddot{d})^2 - (\ddot{d}_x)^2 \Bigr\}\, dt \quad , \tag{45}$$

which, as expected, should be independent of observer angle θ, and the parallel polarization in the beam-observer direction plane is

$$\frac{d\mathcal{E}_{\text{rad}}^{\|}}{d\Omega} \;=\; \cos^2\theta\, \frac{d\mathcal{E}_{\text{rad}}^{\perp}}{d\Omega} + \frac{\sin^2\theta}{2c^3}\int_0^{b_{\max}} b\,db \int_{-\infty}^{\infty} (\ddot{d}_x)^2\, dt \quad , \tag{46}$$

which *contains all the angular dependence*, again as expected.

• Finally, we posit the frequency spectrum in the most efficient manner. Derivation from first principles would proceed along the lines of that above for the temporal integration. Everywhere in place of the temporal integration, one could substitute

$$\int_{-\infty}^{\infty} \bigl(\ddot{\mathbf{d}}\bigr)^2\, dt \;\to\; \int_{-\infty}^{\infty} \bigl(\omega^2 \mathbf{d}_\omega\bigr)^2\, \frac{d\omega}{2\pi} \quad , \tag{47}$$

which is essentially obtained by using **Parseval's relation** for Fourier transforms combined with the second derivative operation in Fourier space. The resolution of components and the azimuthal angle averaging then proceeds as before, and the final result can be quickly written down:

$$\frac{d\mathcal{E}_{\text{rad}}}{d\Omega d\omega} \;=\; \alpha_0(\omega) + \alpha_2(\omega)\, \frac{3\cos^2\theta - 1}{2} \quad , \tag{48}$$

where

$$\begin{aligned} \alpha_0(\omega) \;&=\; \frac{\omega^4}{3\pi c^3 \mathcal{A}} \int_0^{b_{\max}} \bigl(\mathbf{d}_\omega\bigr)^2\, b\,db \\ \alpha_2(\omega) \;&=\; \frac{\omega^4}{6\pi c^3 \mathcal{A}} \int_0^{b_{\max}} \Bigl\{ \bigl(\mathbf{d}_\omega\bigr)^2 - 3\bigl(\mathbf{d}_{x\omega}\bigr)^2 \Bigr\}\, b\,db \quad . \end{aligned} \tag{49}$$

• All the preceding analysis can be adapted for bound Coulomb orbits of high angular momentum, i.e. rosettes, clearly indicating that in classical electrodynamics, such orbits are <u>energetically unstable to radiative decay</u>.