11. DIPOLE RADIATION

Matthew Baring — Lecture Notes for PHYS 532, Spring 2023

1 Electric Dipole Radiation

The focus for now is on radiation from non-relativistic particles. For a particle executing simple harmonic oscillations of frequency $\omega = 2\pi v/\lambda$, the scale of its acceleration is given by $\dot{v} \sim v\,\omega$. The ratio of the radiation to velocity electric (and magnetic) fields can be expressed simply as

$$\frac{|\mathbf{E}_{\rm rad}|}{|\mathbf{E}_{\rm vel}|} \sim \frac{R\,|\dot{\boldsymbol{\beta}}|}{c} \sim \left(\frac{v}{c}\right)^2 \frac{2\pi R}{\lambda} \quad . \tag{1}$$

Clearly, in the near zone $R \lesssim \lambda$, the velocity field dominates the radiation field, whereas in a sufficiently far zone $R \gtrsim \lambda(c/v)^2$ the radiation field dominates so that emission can transpire.

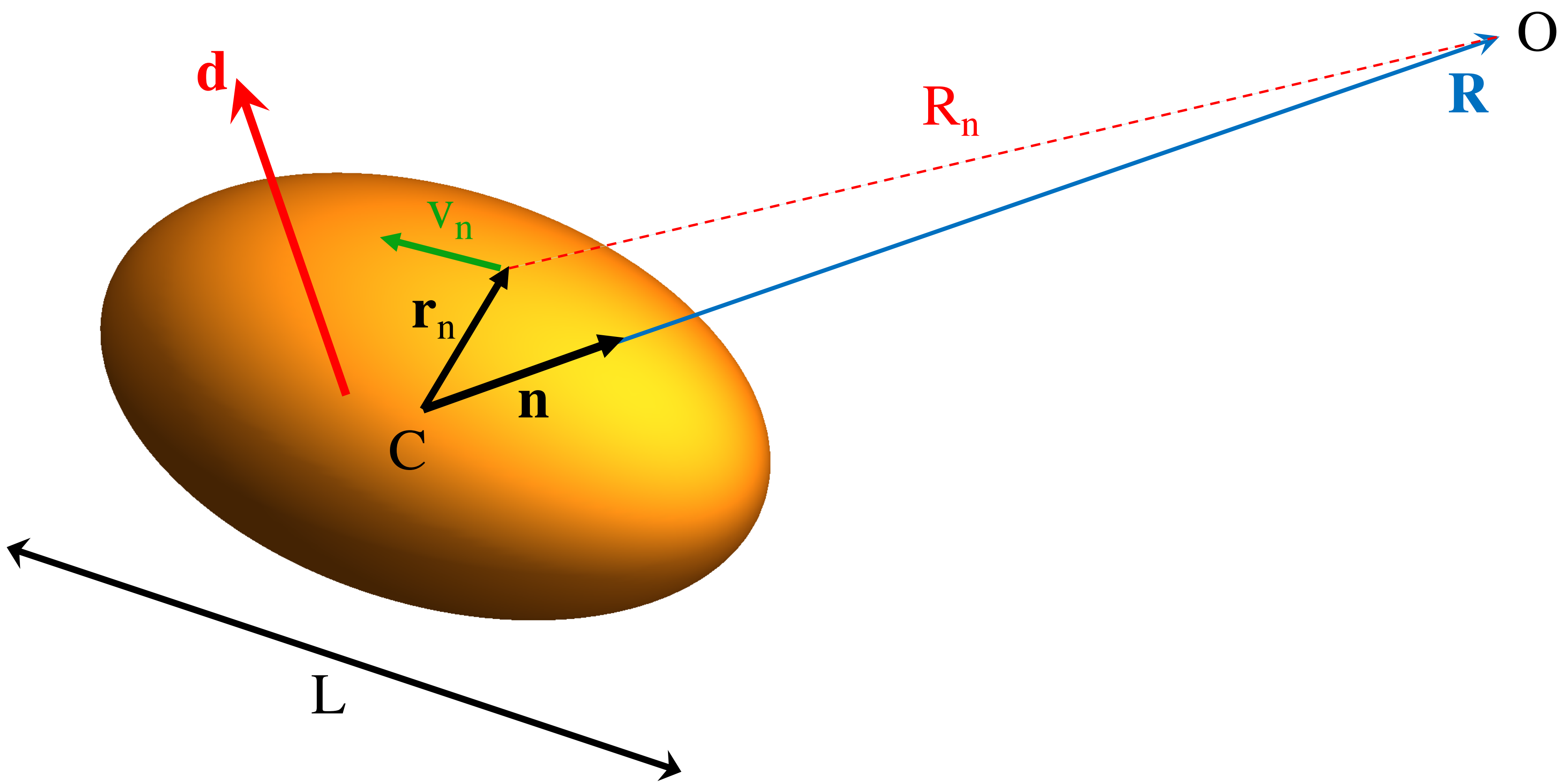
Plot: Dipole Radiation Geometry

- The relative time delays for radiation within our system of charges are on the scale of $\delta t_{\rm em} \sim \mathbf{r}_n \cdot \mathbf{n}/c$. The relative time delays for changes in the charge distribution are on timescales of $\delta t_n \sim |\mathbf{r}_n|/|\mathbf{v}_n|$. For the dipole radiation formalism to be valid, the process of averaging over the charge distribution must be robust, and so $\delta t_{\rm em} \ll \delta t_n$ must hold. Hence, the wavelength $\lambda = 2\pi c/\omega$ of the wave must far exceed the scale L of the system. This can only happen when **L&L Sec. 67**

$$\frac{|\mathbf{v}_n|}{c} \ll 1 \quad : \quad \text{dipole approximation}\ , \tag{2}$$

i.e. *all charges move non-relativistically.*

Dipole Radiation Geometry



1.1 The Larmor Formula

We seek a formula for the total power radiated by an accelerating charge in the far-field zone. The path can start with our vector potential

$$\mathbf{A} \;\approx\; \frac{1}{cR}\sum_n \int \mathbf{j}\left(t - \frac{R}{c} + \frac{\mathbf{r}_n \cdot \mathbf{n}}{c},\, \mathbf{r}_n\right) dV \;\rightarrow\; \frac{1}{cR}\sum_n q_n \mathbf{v}_n \;=\; \frac{\dot{\mathbf{d}}}{cR} \quad . \tag{3}$$

The latter step discretizes the current density using

$$\mathbf{j} \rightarrow \sum_n q_n \mathbf{v}_n \;=\; \frac{d}{dt}\Big(\sum_n q_n \mathbf{r}_n\Big) \;=\; \dot{\mathbf{d}} \quad , \tag{4}$$

and removes the <u>relative</u> time retardation between charges, since it is small in the **dipole radiation approximation**. Accordingly, we explore the <u>coherent</u> radiation contribution. The dipole radiation fields then become

$$\begin{aligned} \mathbf{B}_{\text{rad}} \;&=\; \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \times \mathbf{n} \;=\; \frac{1}{cR}\frac{\partial}{\partial t}\left(\sum_n q_n \boldsymbol{\beta}_n\right) \times \mathbf{n} \;\equiv\; \frac{\ddot{\mathbf{d}} \times \mathbf{n}}{c^2 R} \quad , \\ \mathbf{E}_{\text{rad}} \;&=\; \mathbf{B}_{\text{rad}} \times \mathbf{n} \;=\; \frac{1}{c}\left(\frac{\partial \mathbf{A}}{\partial t} \times \mathbf{n}\right) \times \mathbf{n} \;=\; \frac{(\ddot{\mathbf{d}} \times \mathbf{n}) \times \mathbf{n}}{c^2 R} \quad . \end{aligned} \tag{5}$$

These could equivalently be obtained from our original radiation fields specialized to the $\beta \ll 1$ limit (with a single retarded time):

$$\mathbf{E}_{\text{rad}} \;=\; \frac{q_n}{cR}\left[\mathbf{n} \times \left(\mathbf{n} \times \dot{\boldsymbol{\beta}}_n\right)\right]_{\text{ret}} \quad , \quad \mathbf{B}_{\text{rad}} \;=\; \Big[\mathbf{n} \times \mathbf{E}_{\text{rad}}\Big]_{\text{ret}} \quad . \tag{6}$$

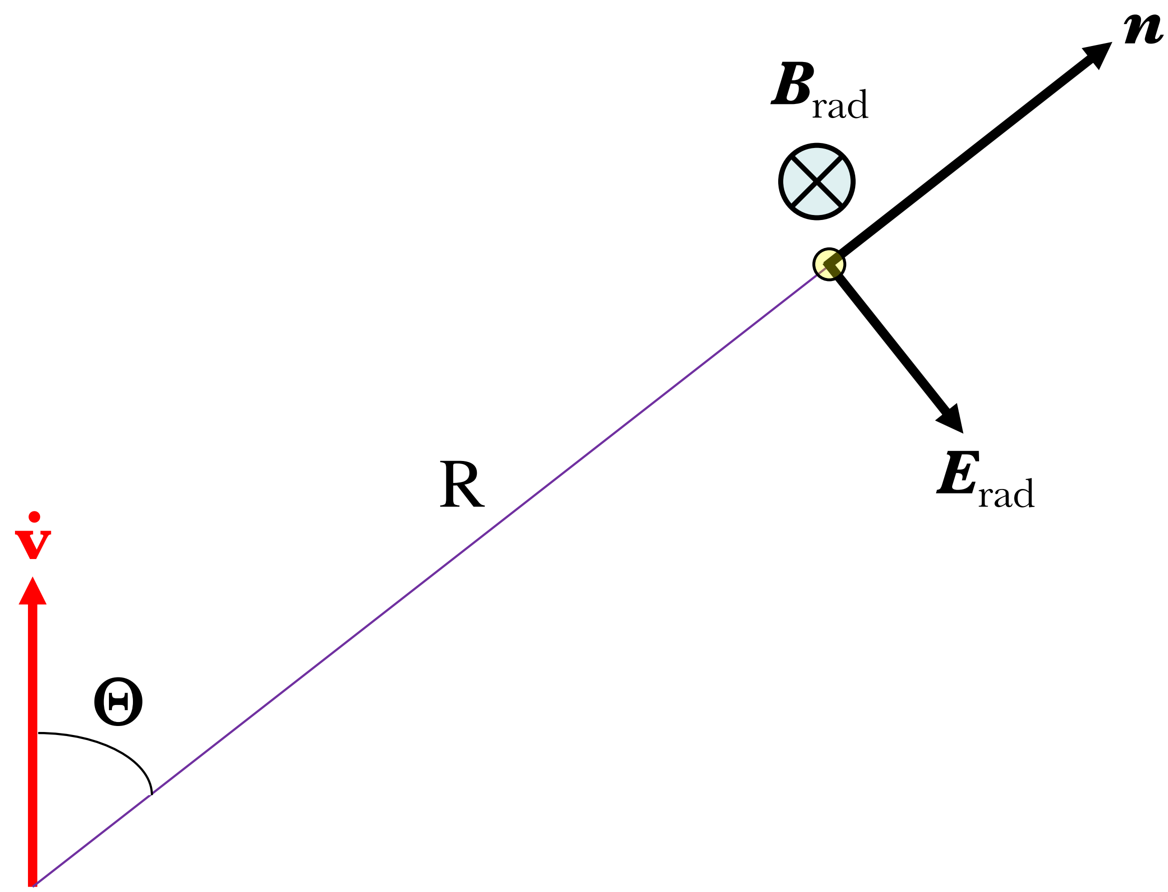
The signature of these radiation results is that the fields are proportional to $\ddot{\mathbf{d}}$; these waves are termed **dipole radiation**.

$\boxed{\textbf{Plot:}}$ Radiation Geometry for a Moving Charge

If Θ is the angle between $\mathbf{n}$ and $\dot{\mathbf{v}}_n$ or $\ddot{\mathbf{d}}$, then the magnitudes of the radiation fields are (using $(\ddot{\mathbf{d}} \times \mathbf{n}) \times \mathbf{n} = \ddot{\mathbf{d}} - (\mathbf{n} \cdot \ddot{\mathbf{d}})\,\mathbf{n}$ for $\mathbf{E}_{\text{rad}}$)

$$\big|\mathbf{E}_{\text{rad}}\big| \;=\; \big|\mathbf{B}_{\text{rad}}\big| \;=\; \left(\sum_n \frac{q_n |\dot{\boldsymbol{\beta}}_n|}{cR}\right) \sin\Theta \;=\; \frac{|\ddot{\mathbf{d}}|}{c^2 R} \sin\Theta \quad . \tag{7}$$

Radiation Geometry for a Moving Charge



- Electric and magnetic radiation field configurations for a slowly moving, accelerating charge. The direction of $\boldsymbol{B}_{\mathrm{rad}}$ is into the page.

This uses the result

$$\left[\ddot{\mathbf{d}} - (\mathbf{n}\cdot\ddot{\mathbf{d}})\,\mathbf{n}\right]^2 \;=\; \left(\ddot{d}\right)^2\left[1 - 2\cos^2\Theta + \cos^2\Theta\right] \;=\; \left(\ddot{d}\right)^2\sin^2\Theta \quad . \tag{8}$$

The Poynting vector flux in the direction of $\mathbf{n}$ is then

$$S \;=\; \frac{c}{8\pi}\left\{\left|\mathbf{E}_{\rm rad}\right|^2 + \left|\mathbf{B}_{\rm rad}\right|^2\right\} \;=\; \frac{c}{4\pi}\,\frac{|\ddot{\mathbf{d}}|^2}{c^4R^2}\,\sin^2\Theta \quad . \tag{9}$$

This is a classic **dipole radiation formula** with its $\sin^2\Theta$ angular dependence, and no power along the direction of $\ddot{\mathbf{d}}$ for transverse E/M waves. This is the **antenna pattern** (*toroid*) for classical processes like Thomson scattering, considered shortly. It applies only to non-relativistic charges.

Plot: Radiation Toroidal Antenna Pattern

• Note that the Poynting flux obeys the inverse square law: for the radiated electromagnetic wave(s), energy is conserved in propagating to infinity.

The Poynting flux is the energy per unit time per unit area, so that multiplying it by the area element $dA = R^2 d\Omega$ derives the total radiated power:

$$P \;\equiv\; \frac{d\mathcal{E}}{dt} \;=\; \frac{|\ddot{\mathbf{d}}|^2}{4\pi c^3}\int \sin^2\Theta\, d\Omega \quad . \tag{10}$$

The integral is $8\pi/3$ and the result is **Larmor's formula** for emission from a single accelerating charge q or a charge ensemble:

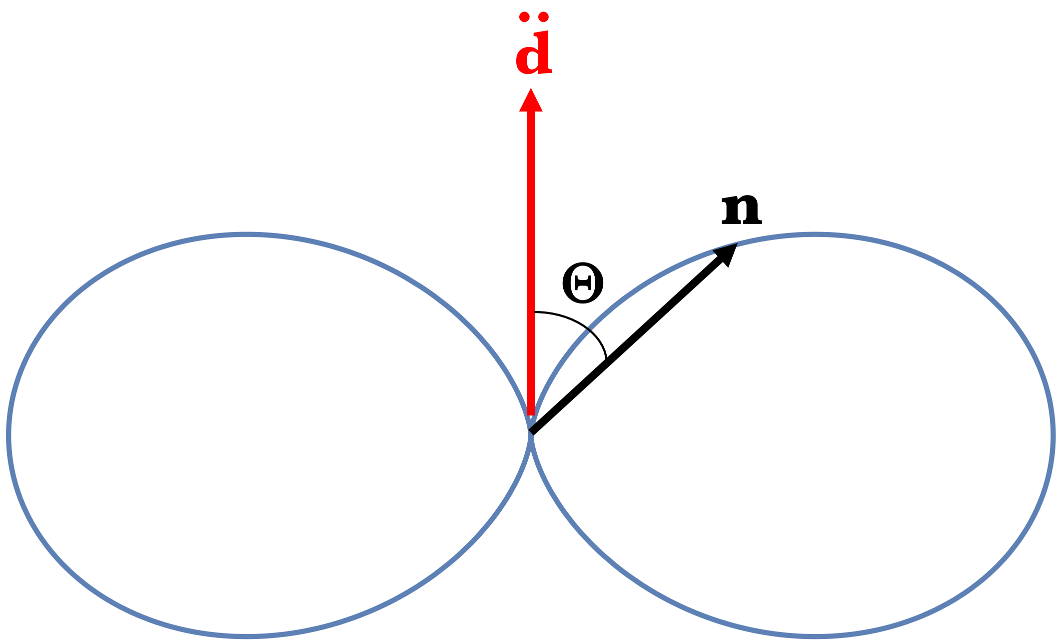
$$P \;=\; \frac{2q^2\dot{\beta}^2}{3c} \;\to\; \frac{2\,|\ddot{\mathbf{d}}|^2}{3c^3} \quad , \tag{11}$$

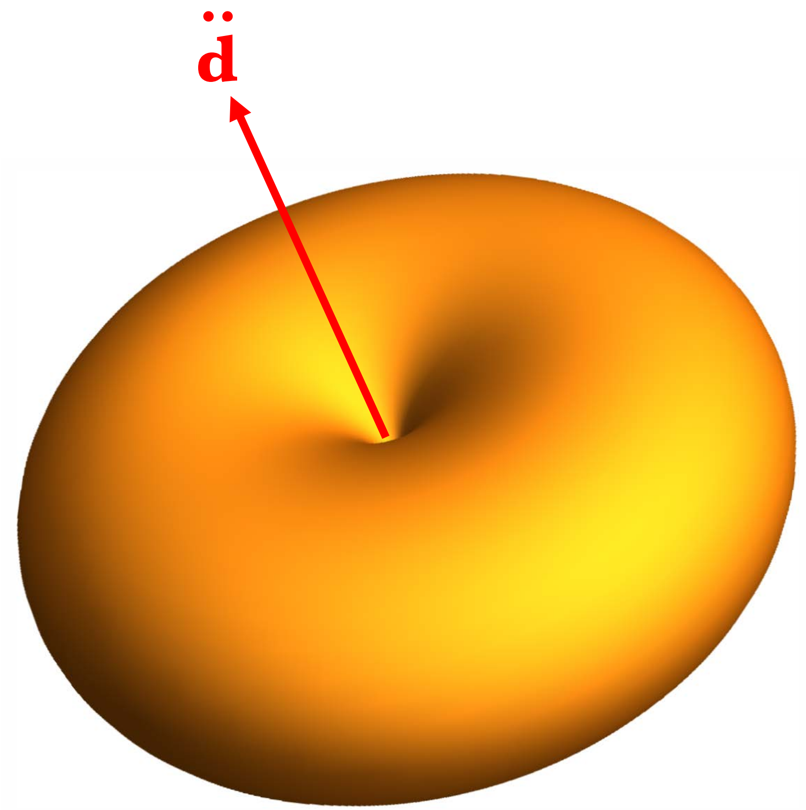
since $|\ddot{\mathbf{d}}| = q\dot{\beta}c$. Observe that the power emitted is proportional to the square of the charge, and the square of the acceleration.

$*$ The direction of $\mathbf{E}_{\rm rad}$ is controlled by $\mathbf{n}$ and $\dot{\boldsymbol{\beta}}$. If the acceleration is linear, then the radiation is 100% polarized in the plane of $\dot{\boldsymbol{\beta}}$ (or $\ddot{\mathbf{d}}$) and $\mathbf{n}$.

• For a system of identical particles under no external force, the dipole moment is proportional to the total mass, and its time derivative is proportional to the velocity of the centre of mass, which does not accelerate. Therefore, the system has $\ddot{\mathbf{d}} = \mathbf{0}$ and cannot radiate.

Radiation Toroidal Antenna Pattern





- Antenna emission pattern of dipole radiation of transverse waves.
- Left is $\sin^2\Theta$ cross section, and right is the 3D toroidal representation, axi-symmetric about the acceleration axis.

The power spectrum of radiation is now simply obtained by inserting the Fourier form of Eq. (5), namely

$$\mathbf{B}_\omega \;=\; \frac{\ddot{\mathbf{d}}_\omega \times \mathbf{n}}{c^2 R} \quad \text{with} \quad \ddot{\mathbf{d}}_\omega \;=\; \int e^{i\omega t}\, \ddot{\mathbf{d}}\, dt \;=\; -\omega^2 \mathbf{d}_\omega \quad , \tag{12}$$

into the general expression for the Fourier power, i.e. radiation spectrum. The result is

$$\frac{d\mathcal{E}_{\rm rad}}{d\omega\, d\Omega\, dt} \;=\; \frac{\omega^4}{2\pi c^3} \left| \mathbf{d}_\omega \times \mathbf{n} \right|^2 \;=\; \frac{\omega^4}{2\pi c^3} \left| \mathbf{d}_\omega \right|^2 \sin^2 \Theta \quad , \tag{13}$$

which integrates over solid angles to give

$$\frac{d\mathcal{E}_{\rm rad}}{d\omega\, dt} \;=\; \frac{4\omega^4}{3c^3} \left| \mathbf{d}_\omega \right|^2 \quad . \tag{14}$$

Thus, if a system of charges is jostled by an incoming E/M wave with a relatively low frequency ω_0, the dipole moment will be $\mathbf{d} \propto \exp\{i\omega_0 t\}$ and then $\mathbf{d}_\omega \sim$ const. The system then re-emits power $\propto \omega^4$; this is the character of **Rayleigh scattering**.

$*$ This is precisely the form of scattering pertinent to sunlight that is incident on the molecules in the sky. Accordingly, blue light is scattered more, and thus the sky appears blue and not red.

2 Dipole Radiation Processes

2.1 Cyclotron Radiation

Consider an electric dipole that rotates periodically in a plane with constant angular rate ω_c, i.e. about the x-axis. Suppose that the magnitude of the dipole is held constant. Then for motion in the $(y,\, z)$-plane,

$$d_y \;=\; d\,\cos\omega_c t \quad , \quad d_z \;=\; d\,\sin\omega_c t \quad . \tag{15}$$

The dipole moment therefore yields $\ddot{\mathbf{d}} = -\omega_c^2\mathbf{d} = -\omega_c^2 d\,(0,\, \cos\omega_c t,\, \sin\omega_c t)$. The power is proportional to $(\ddot{\mathbf{d}}\times\mathbf{n})^2$, which varies with time. Accordingly, a time average (over the period $2\pi/\omega_c$) needs to be taken to provide a suitable measure of the power. Then, if the observer is in the $(x,\, y)$-plane, then $\mathbf{n} = (\sin\Theta,\, \cos\Theta, 0)$, and

$$\begin{aligned} \frac{1}{\omega_c^4}\bigl\langle(\ddot{\mathbf{d}}\times\mathbf{n})^2\bigr\rangle_t \;&=\; \bigl\langle(\mathbf{d}\times\mathbf{n})^2\bigr\rangle_t \;=\; \bigl\langle(\mathbf{d})^2\bigr\rangle_t - \bigl\langle(\mathbf{n}\cdot\mathbf{d})^2\bigr\rangle_t \\ &=\; d^2 - \bigl\langle(d\cos\Theta\cos\omega_c t)^2\bigr\rangle_t \\ &=\; d^2\Bigl(1 - \frac{1}{2}\,\cos^2\Theta\Bigr) \;=\; \frac{d^2}{2}\,\bigl(1+\sin^2\Theta\bigr) \quad . \end{aligned} \tag{16}$$

The time-averaged differential power and solid angle-integrated power are[1]

$$\frac{d\mathcal{E}_{\rm rad}}{d\Omega\, dt} \;=\; \frac{\omega_c^4 d^2}{8\pi c^3}\,\bigl(1+\sin^2\Theta\bigr) \quad\Rightarrow\quad \frac{d\mathcal{E}_{\rm rad}}{dt} \;=\; \frac{2\omega_c^4 d^2}{3c^3} \quad , \tag{17}$$

and *the spectrum is concentrated at frequency* ω_c. Of particular interest is the case where the dipole is formed by a single charge gyrating non-relativistically in a uniform magnetic field $\mathbf{B}$ at the **cyclotron frequency** $\omega_c = qB/mc$. Since $d \equiv qr_g = m\beta_\perp c^2/B$ for gyromotions, the emission is then of power

$$\frac{d\mathcal{E}_{\rm rad}}{d\omega\, dt} \;=\; \frac{d\mathcal{E}_{\rm rad}}{dt}\,\delta(\omega-\omega_c) \;=\; \frac{2}{3}\left(\frac{q^2}{mc^2}\right)^2 \beta_\perp^2 c\, B^2\,\delta(\omega-\omega_c) \quad , \tag{18}$$

and is called **cyclotron radiation**. Here $\beta_\perp c$ is the gyrational speed, and $r_q = q^2/mc^2$ is termed the **classical electrodynamic radius** of the charge.

[1] Remember that here Θ is the viewing angle relative to the plane of gyration [contrasting the L&L choice], i.e., $\mathbf{n} = (\sin\Theta,\, \cos\Theta,\, 0)$, so that $d\Omega = 2\pi\, d(\sin\Theta)$.

2.2 Induced Radiation: Thomson Scattering

The dipole radiation formalism can be applied to charge motions that are driven by incoming electromagnetic waves. This is the domain of scattering. **L&L Sec. 78**

• Consider a 100% linearly-polarized electromagnetic wave that jostles a free charge at rest. The electric field of the wave is in the (y, z)-plane:

$$\mathbf{E} \;=\; \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \alpha) \quad . \tag{19}$$

Thus the $\mathbf{k}$ vector is in the x-direction. The alternating electric field drives the charge in an oscillation. If the charge moves non-relativistically, then the magnetic field contribution to the Lorentz force equation can be neglected. Then the motion is a 1D oscillation in the direction of field vector $\mathbf{E}_0$, and the equation of motion can be simply solved as an inhomogeneous ODE:

$$m\,\ddot{\mathbf{r}} \;=\; q\,\mathbf{E} \quad\Rightarrow\quad \mathbf{r} \;=\; -\frac{q}{m\omega^2}\,\mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \alpha) \quad , \tag{20}$$

motion presumed to take place near $\mathbf{r} = \mathbf{0}$. The maximum speed of the induced oscillation is then $qE_0/(m\omega)$. This must be much less than c, so the non-relativistic criterion for the applicability of the dipole approximation is

$$\left(\frac{v_{\rm max}}{c}\right)^2 \;\approx\; \left(\frac{qE_0}{m\omega c}\right)^2 \;=\; 4\pi\,\frac{q^2}{mc^2}\,\frac{U_{\rm wave}}{m\omega^2} \;\ll\; 1 \quad . \tag{21}$$

This bounds the intensity of the incoming wave, and is usually not very restrictive in either terrestrial or astronomical settings: i.e. the dipole formalism is widely applicable to this scattering problem.

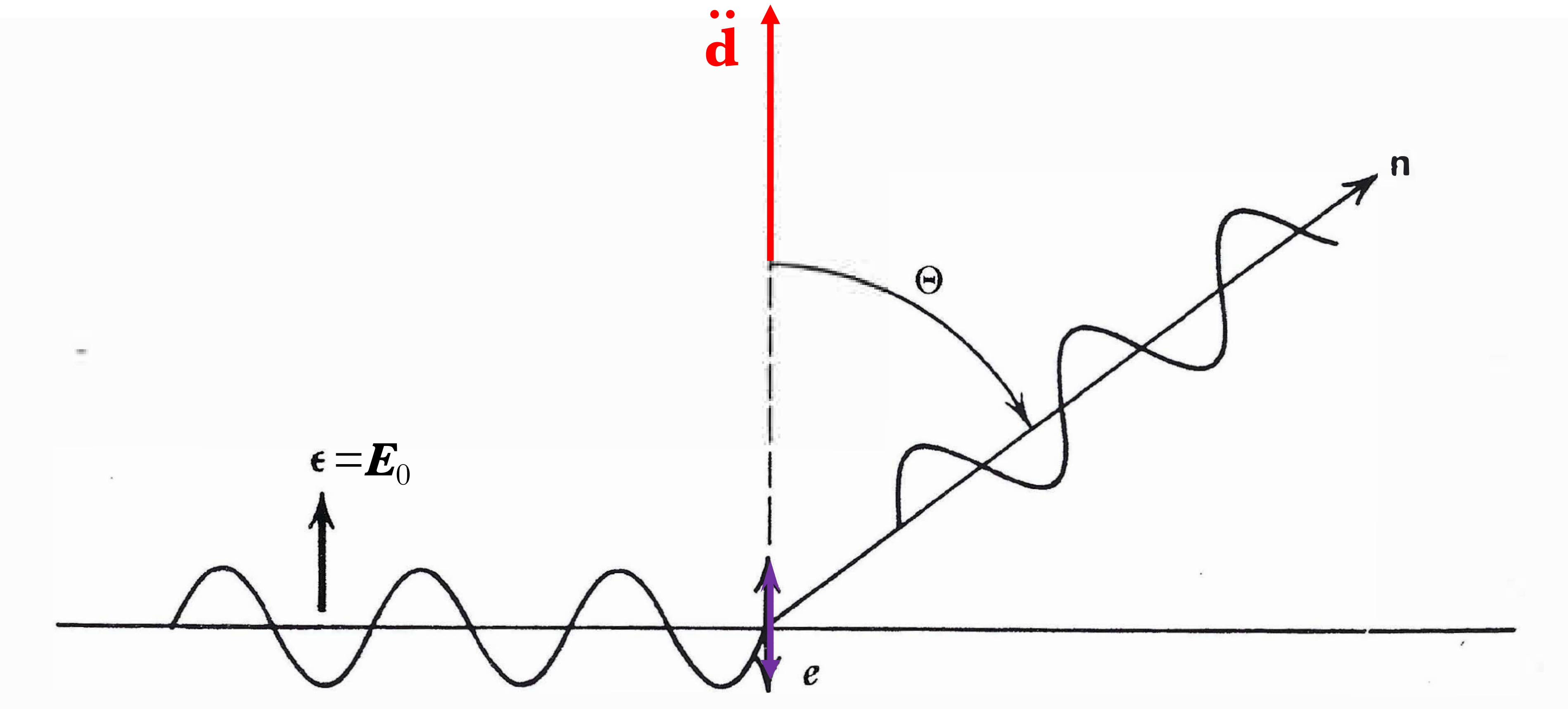
Assuming this is satisfied, then the amplitude of the oscillation is also small, sufficiently so that $|\mathbf{k}\cdot\mathbf{r}| \lesssim \omega r_{\rm max}/c = qE_0/(m\omega c) \ll 1 \lesssim \omega t$, and the displacement of the charge in influencing the wave generation can be neglected. The dipole moment of the charge satisfies

$$\ddot{\mathbf{d}} \;=\; \frac{q^2}{m}\,\mathbf{E} \quad ; \tag{22}$$

the charge thus radiates, and the process is a scattering of the E/M wave.

Plot: Thomson Scattering Geometry

Thomson Scattering Geometry



- Scattering of linearly-polarized radiation by an electron.
- Fig. 3.6 of Rybicki & Lightman: *Radiative Processes in Astrophysics* (1979).