2 Fields of Moving Charges: Large Distances

It is time to treat contained systems of charges that are in motion and therefore can radiate electromagnetic waves. So far, we have isolated how charges respond to fields, and create fields, and also how electromagnetic waves can exist. To close the loop of connectivity between charges and fields, we explore how charge motions can generate E/M waves: **true electrodynamics**. **L&L Sec. 66**

Plot: Radiation Geometry for a Charge Ensemble

Consider an ensemble of charges distributed within a small, finite volume $V = d^3\mathbf{r}$ of dimensions less than some scale L. Let the origin C of coordinates be contained within the volume, and the position of each charge be denoted by $\mathbf{r}_n$, with $|\mathbf{r}_n| \leq L$. Consider a distant observation point O at position $\mathbf{R} = R\,\mathbf{n}$ such that $R \gg L$. At such large distances the potentials will receive contributions from each charge according to its distance

$$R_n \;=\; |\mathbf{R} - \mathbf{r}_n| \;\approx\; R - \mathbf{r}_n \cdot \mathbf{n} \tag{23}$$

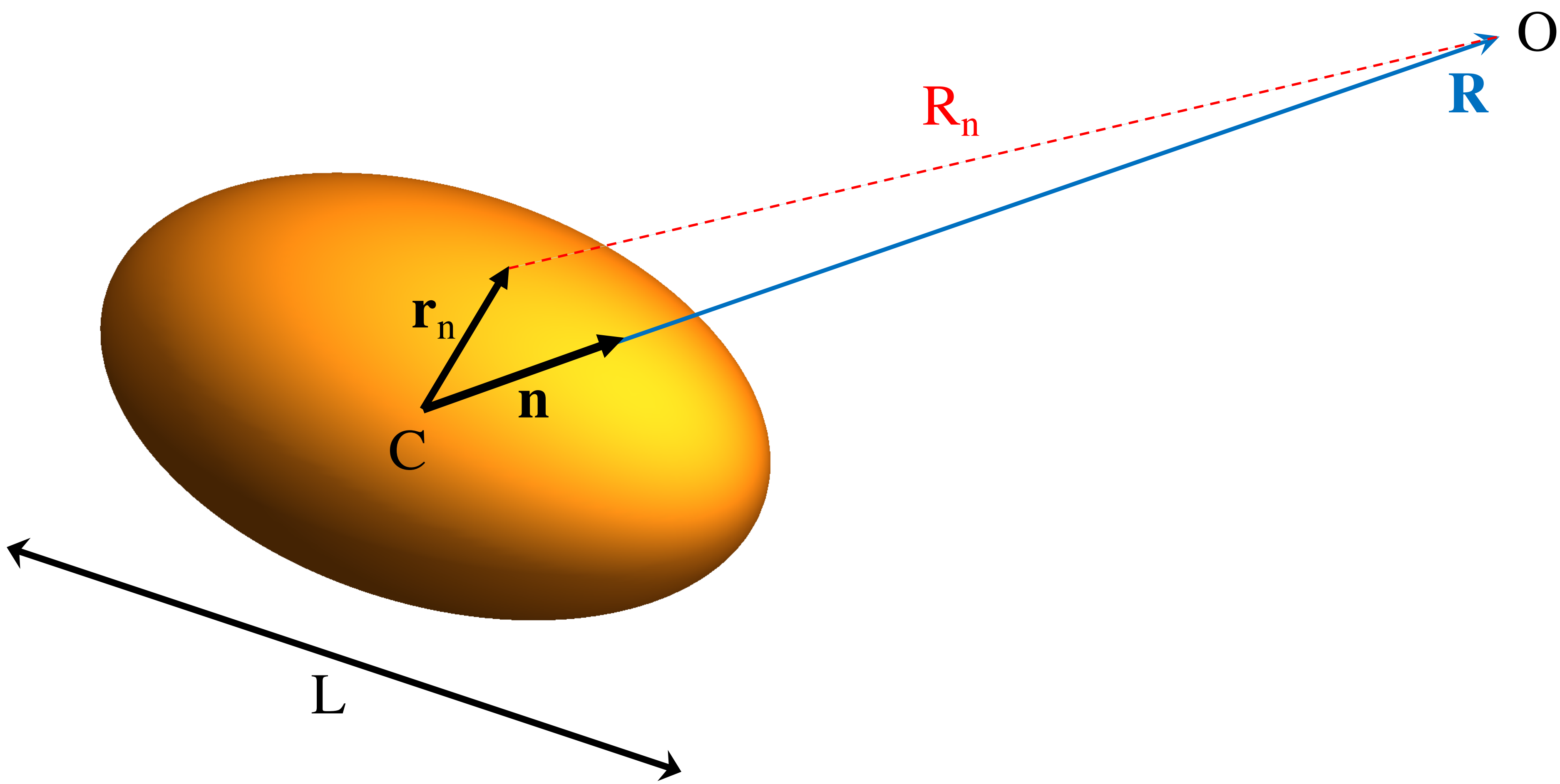
from the observation point. This Taylor series expansion to leading order simplifies the retarded potentials. Specifically, the volume integrations do not depend on factors of R, and so we have two retarded potentials of the approximate form:

$$\begin{aligned} \phi \;&=\; \sum_n \phi_n(t - R_n/c) \;\approx\; \frac{1}{R}\sum_n \int \rho\left(t - \frac{R}{c} + \frac{\mathbf{r}_n\cdot\mathbf{n}}{c},\, \mathbf{r}_n\right) dV \\ \mathbf{A} \;&=\; \sum_n \mathbf{A}_n(t - R_n/c) \;\approx\; \frac{1}{cR}\sum_n \int \mathbf{j}\left(t - \frac{R}{c} + \frac{\mathbf{r}_n\cdot\mathbf{n}}{c},\, \mathbf{r}_n\right) dV \quad . \end{aligned} \tag{24}$$

Here the summation over q denotes the summation over the contribution of each discrete charge. In general, the positions $\mathbf{r}_n$ are determined at the individual retarded time $\tau_n = t - R_n/c$. However, for this **far-field** approximation, these are computed at the common retarded time $t - R/c$, thereby incurring a small error that is second order in $\mathbf{r}_n/c$.

- This approximation is only valid if $\lambda \gg L$, i.e. the typical wavelength of the emitted wave far exceeds the size of the charge system.

Radiation Geometry for a Charge Ensemble



Observe that for $\tau = t - R/c$ as the "mean" propagation (signal) time of electromagnetic information out to the observation point, the forms of these potentials are

$$A^\mu \;=\; A^\mu\bigl(c\tau + \mathbf{r}_n \cdot \mathbf{n}\bigr) \quad , \tag{25}$$

which is precisely the form of a solution to the wave equation in the variables $(c\tau,\, x = R)$. Thus, *the far-field configuration is an electromagnetic wave*, of non-zero amplitude under suitable conditions to be identified.

- Given that we know we have a plane wave, we can identify its key characteristics. First, its propagation direction is parallel to $\mathbf{n}$, out to the observer. Next, the wave solutions are transverse waves, so that $\mathbf{E} \cdot \mathbf{n} = 0 = \mathbf{B} \cdot \mathbf{n}$. Moreover, the electric and magnetic fields are orthogonal to each other, so $\mathbf{E} = \mathbf{B} \times \mathbf{n}$. We have already established these fields in terms of the vector potential $\mathbf{A}$, and so write

$$\mathbf{B}_{\rm rad} \;=\; \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \times \mathbf{n} \quad , \quad \mathbf{E}_{\rm rad} \;=\; \frac{1}{c}\left(\frac{\partial \mathbf{A}}{\partial t} \times \mathbf{n}\right) \times \mathbf{n} \quad . \tag{26}$$

Here the subscript labels "rad" are introduced to designate the radiation fields in the **wave zone**. If we presume that $\mathbf{v}_n = $const., then the vector current $\mathbf{j}$ is constant in time, and thus so also is $\mathbf{A}$ using Eq. (24). Then it is deduced that $\mathbf{B}_{\rm rad} = \mathbf{0} = \mathbf{E}_{\rm rad}$, and there is no electromagnetic radiation to infinity. Accordingly we arrive at a fundamental feature of electrodynamics: *charges must accelerate in order to radiate electromagnetic waves*.

As the vector potential in Eq. (25) satisfies the wave equation in Eq. (2), the time derivative scales with R/c [see also Eq. (24)], so that it follows that

$$|\mathbf{B}_{\rm rad}| \;\sim\; \frac{1}{R} \quad , \quad |\mathbf{E}_{\rm rad}| \;\sim\; \frac{1}{R} \quad \Rightarrow \quad U_{\rm em} \;\sim\; \frac{1}{R^2} \tag{27}$$

at larger distances from the charge ensemble. It follows that the energy density obeys the inverse square law, signifying *conservation of energy flux in spherical geometry*. The flux, of course, is given by the Poynting vector:

$$\mathbf{S} \;=\; \frac{c}{4\pi}\,|\mathbf{B}_{\rm rad}|^2\,\mathbf{n} \quad \Rightarrow \quad dE \;=\; \frac{c}{4\pi}\,|\mathbf{B}_{\rm rad}|^2\,R^2\,d\Omega\,dt \quad . \tag{28}$$

The conservation law then follows, though it is subject to the implicit assumption that time variations on the scale L/c of the charge ensemble do not differ much with distance R.

• Notation comment: the notation used here differs from that employed in L&L, with the following correspondences: $\mathbf{R} \to \mathbf{R}_0$, $\mathbf{r}_n \to \mathbf{r}$ and $\mathbf{R}_n \to \mathbf{R}$, the latter variables in each pair being adopted in the textbook.

2.1 Spectral Representation of the Fields

Now we derive the Fourier transforms of the fields, which are needed to eventually *define the radiation spectrum*. These are established using the Landau & Lifshitz normalization for Fourier transforms,[1]

$$\mathbf{A}(t) \;=\; \int e^{-i\omega t}\,\mathbf{A}_\omega\,\frac{d\omega}{2\pi} \quad\Rightarrow\quad \mathbf{A}_\omega \;=\; \int e^{i\omega t}\,\mathbf{A}(t)\,dt \quad . \tag{29}$$

Into this we insert the retarded potential form for the vector potential from the charge/current ensemble in Eq. (24):

$$\mathbf{A}_\omega \;=\; \frac{1}{cR}\sum_n \int e^{i\omega t}\int \mathbf{j}\left(t-\frac{R}{c}+\frac{\mathbf{r}_n\cdot\mathbf{n}}{c},\,\mathbf{r}_n\right)\,dV\,dt \tag{30}$$

Performing the temporal integration first, we arrive at the Fourier component of the three-current with a “phase-shift” factor due to the change of variables in the time integral (this is a signal propagation factor):

$$\int e^{i\omega t}\mathbf{j}\left(t-\frac{R}{c}+\frac{\mathbf{r}_n\cdot\mathbf{n}}{c},\,\mathbf{r}_n\right)\,dt \;=\; \mathbf{j}_\omega\,e^{ikR-i\mathbf{k}\cdot\mathbf{r}_n} \quad . \tag{31}$$

Here we have introduced the wavevector $\mathbf{k}$ such that $\mathbf{k}=k\,\mathbf{n}$ and $|\mathbf{k}|=\omega/c$. It follows that

$$\mathbf{A}_\omega \;=\; \frac{e^{ikR}}{cR}\sum_n \int \mathbf{j}_\omega\,e^{-i\mathbf{k}\cdot\mathbf{r}_n}\,dV \tag{32}$$

If we discretize the charges, then the currents become $\mathbf{j}_n \to q_n\mathbf{v}_n(t)$, and

$$\begin{aligned}\mathbf{A}_\omega \;&\to\; \frac{e^{ikR}}{cR}\sum_n \int q_n\mathbf{v}_n\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r}_n)}\,dt \\ &\equiv\; \frac{e^{ikR}}{cR}\sum_n q_n\int e^{i(\omega t-\mathbf{k}\cdot\mathbf{r}_n)}\,d\mathbf{r}_n \quad ,\end{aligned} \tag{33}$$

with $\mathbf{v}_n dt \to d\mathbf{r}_n$. In the last integral, we implicitly assume that $t=t(\mathbf{r}_n)$.

[1] This is technically the correct definition of a Fourier transform – note that the sign of the complex exponential is switched in the spatial Fourier resolution for electrostatics: this is done deliberately so that wave character $\omega t-\mathbf{k}\cdot\mathbf{r}$ is preserved.

Using Eq. (26) we can determine the Fourier components of the radiation fields via the manipulation

$$\mathbf{B}_\omega \;=\; \int e^{i\omega t}\,\mathbf{B}\,dt \;=\; \frac{1}{c}\int dt\,e^{i\omega t}\left(\frac{\partial \mathbf{A}}{\partial t}\times\mathbf{n}\right) \;=\; -\frac{i\omega}{c}\,\mathbf{A}_\omega\times\mathbf{n} \quad , \tag{34}$$

and likewise for the electric field. Thus, since $\mathbf{n}=\mathbf{k}/k$,

$$\mathbf{B}_\omega \;=\; i\mathbf{k}\times\mathbf{A}_\omega \quad , \quad \mathbf{E}_\omega \;=\; \mathbf{B}_\omega\times\mathbf{n} \;=\; \frac{i}{k}\,(\mathbf{k}\times\mathbf{A}_\omega)\times\mathbf{k} \quad . \tag{35}$$

Observe that in Fourier space, curls map over to cross products. Inserting the expression in Eq. (33) for the vector potential, we therefore have

$$\begin{aligned} \mathbf{B}_\omega \;&=\; i\,\frac{e^{ikR}}{cR}\sum_n q_n\int e^{i(\omega t-\mathbf{k}\cdot\mathbf{r}_n)}\,\mathbf{k}\times d\mathbf{r}_n \quad , \\ \mathbf{E}_\omega \;&=\; -\frac{i}{k}\,\frac{e^{ikR}}{cR}\sum_n q_n\int e^{i(\omega t-\mathbf{k}\cdot\mathbf{r}_n)}\,\mathbf{k}\times\big(\mathbf{k}\times d\mathbf{r}_n\big) \quad . \end{aligned} \tag{36}$$

The triple vector product can be expanded in terms of scalar products, and sometimes this step is useful. Remember that $t=t(\mathbf{r}_n)$ is implicit.

• The spectral power of radiation to large distances can be obtained as follows. The Poynting flux is $(c/4\pi)B^2 = (c/4\pi)E^2$, which amounts to the *energy per unit time per unit area* integrated over the period or 2π in angular frequency ω . Hence we multiply this by $d\omega/2\pi$ to form the Fourier transform. However, we are expressing it terms of the Fourier components of $\mathbf{B}_\omega$ and so we including *a factor of two for the squaring in frequency space.* Therefore, introducing the area element $R^2 d\Omega$ of a sphere,

$$\frac{d\mathcal{E}_{\rm rad}}{d\omega\,d\Omega\,dt} \;=\; \frac{c}{2\pi}\,\big|\mathbf{B}_\omega\big|^2\,R^2 \quad . \tag{37}$$

Again, this evinces the character of the inverse square law, signifying energy conservation. This form will be used to specify the radiation spectral forms for different mechanisms below, specifically Rayleigh and Thomson scattering, and cyclotron radiation.

3 Radiation Fields at Near Distances

There are many cases where one needs to know the radiation field at near distances. The situation explored here is where the distance scale R is comparable to the wavelength λ of the emitted radiation, so that the wave is no longer quasi-planar. Yet, these scales are still much greater than that (L) of the charge configuration. For non-relativistic charges, *the dipole formula for the vector potential still applies*:

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$$\mathbf{A} \;=\; \frac{1}{cR}\sum_n \int \mathbf{j}\,(\tau_n,\, \mathbf{r}_n)\; dV \;\to\; \frac{1}{cR}\sum_n q_n \mathbf{u}_n \;\to\; \frac{\dot{\mathbf{d}}}{cR} \quad . \tag{38}$$

The radiation still obeys the wave equation, but the spatial derivatives now must reflect the curvature of the wavefront. The magnetic field is routinely computed:

$$\mathbf{B} \;=\; \nabla\times\mathbf{A} \;=\; \frac{1}{c}\,\nabla\times\left(\frac{\dot{\mathbf{d}}}{R}\right) \quad . \tag{39}$$

Computing the electric field is a little more involved since it involves a mixture of time and space derivatives that no longer permit the employment of cross products appropriate for planar waves. Inserting the vector potential form into the Lorenz gauge condition gives

$$\nabla\cdot\mathbf{A} + \frac{1}{c}\,\frac{\partial\phi}{\partial t} \;=\; 0 \quad\Rightarrow\quad \phi \;=\; -\nabla\cdot\left(\frac{\mathbf{d}}{R}\right) \quad . \tag{40}$$

Remember that throughout, the dipole moment must be evaluated at the retarded time $t - R/c$ common to the charge ensemble. Therefore one can determine the electric field:

$$\mathbf{E} \;=\; -\nabla\phi - \frac{1}{c}\,\frac{\partial\mathbf{A}}{\partial t} \;=\; \nabla\left[\nabla\cdot\left(\frac{\mathbf{d}}{R}\right)\right] - \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}\left(\frac{\mathbf{d}}{R}\right) \quad . \tag{41}$$

Here the time variations are greatest within the charge ensemble, on the scale of L, not on the scale of R, and so the time derivative is applied to $\mathbf{d}$ only. Since $\mathbf{A}$ obeys the wave equation, then so does $\dot{\mathbf{d}}/R$ and also $\mathbf{d}/R$. This affords the opportunity to replace the time derivatives with space ones:

$$\mathbf{E} \;=\; \nabla\left[\nabla\cdot\left(\frac{\mathbf{d}}{R}\right)\right] - \nabla^2\left(\frac{\mathbf{d}}{R}\right) \;=\; \nabla\times\left[\nabla\times\left(\frac{\mathbf{d}}{R}\right)\right] \quad , \tag{42}$$

where we have employed a standard vector identity to render the vector algebra more compact. The $\mathbf{d}$ factors <u>must</u> remain inside the derivatives.

11. DIPOLE RADIATION

Matthew Baring — Lecture Notes for PHYS 532, Spring 2023

1 Dipole Radiation

The focus for now is on radiation from non-relativistic particles. For a particle executing simple harmonic oscillations of frequency $\omega = 2\pi v/\lambda$, the scale of its acceleration is given by $\dot{v} \sim v\,\omega$. The ratio of the radiation to velocity electric (and magnetic) fields can be expressed simply as

$$\frac{|\mathbf{E}_{\rm rad}|}{|\mathbf{E}_{\rm vel}|} \;\sim\; \frac{R\,|\dot{\boldsymbol{\beta}}|}{c} \;\sim\; \left(\frac{v}{c}\right)^2 \frac{2\pi R}{\lambda} \quad . \tag{1}$$

Clearly, in the near zone $R \lesssim \lambda$, the velocity field dominates the radiation field, whereas in a sufficiently far zone $R \gtrsim \lambda(c/v)^2$ the radiation field dominates so that emission can transpire.

$\boxed{\textbf{Plot:}}$ Dipole Radiation Geometry

• The relative time delays for radiation within our system of charges are on the scale of $\delta t_{\rm em} \sim \mathbf{r}_n \cdot \mathbf{n}/c$. The relative time delays for changes in the charge distribution are on timescales of $\delta t_n \sim |\mathbf{r}_n|/|\mathbf{v}_n|$. For the dipole radiation formalism to be valid, the process of averaging over the charge distribution must be robust, and so $\delta t_{\rm em} \ll \delta t_n$ must hold. Hence, the wavelength $\lambda = 2\pi c/\omega$ of the wave must far exceed the scale a L of the system. This can only happen when

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$$\frac{|\mathbf{v}_n|}{c} \;\ll\; 1 \quad : \quad \text{dipole approximation}\ , \tag{2}$$

i.e. *all charges move non-relativistically.*