10. RADIATION OF ELECTROMAGNETIC WAVES

Matthew Baring — Lecture Notes for PHYS 532, Spring 2023

1 Retarded Potentials from Moving Charges

The final thrust of the course is to explore how moving charges generate electromagnetic radiation, i.e. set light on its way. This requires a return to Maxwell's equations in their entirety, and construction of solutions for the potentials that incorporate the constraints of causality.

1.1 Retarded Potentials

The focus here is to extend the wave equation formalism to incorporate the influence of charges and currents in producing electromagnetic fields. The starting point is, as always, Maxwell's equations, specifically those in inhomogeneous form: **L&L Sec. 62**

$$-\frac{4\pi}{c}\, j^\mu \;=\; \frac{\partial F^{\mu\nu}}{\partial x^\nu} \;=\; \frac{\partial}{\partial x^\nu}\left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}\right) \;=\; \cancel{\frac{\partial}{\partial x_\mu}\frac{\partial A^\nu}{\partial x^\nu}} - \frac{\partial^2 A^\mu}{\partial x^\nu \partial x_\nu} \quad . \tag{1}$$

As before, we invoke the Lorentz gauge to eliminate one of the terms. Thus, the field changes couple to time-varying four-currents. The path to the solution is just as for the vacuum developments for the wave equation, and so four inhomogeneous wave equations result:

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} \;=\; -\frac{4\pi}{c}\,\mathbf{j} \quad , \quad \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \;=\; -4\pi\rho \quad . \tag{2}$$

Here $\rho = \rho(t,\, \mathbf{r})$ and $\mathbf{j} = \mathbf{j}(t,\, \mathbf{r})$. The protocols for solving for electrostatics and magnetostatics suggest exploring the Coulomb case first (scalar potential) for a point charge density at the origin; this serves as the Green's function for the problem, with $\rho \propto \delta(\mathbf{r})\, \sigma(t)$. The spherical symmetry of the source function suggests writing the Laplacian operator in spherical coordinates. Hence, for true isotropy,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r} \right) \;\equiv\; \frac{1}{r^2}\frac{\partial^2 (r\phi)}{\partial r^2} \;=\; \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - 4\pi\rho \quad . \tag{3}$$

When $\rho = 0$, this is just the wave equation for spherical fronts.

It follows that the function $r\phi$ solves the wave equation in radial space, i.e.,

$$r\phi \;\to\; f_1(t - r/c) + f_2(t + r/c) \quad , \tag{4}$$

for any functions f_i. The f_2 function can be rejected on physical grounds, namely that the wave must propagate outward from a point. It then follows that the potential satisfies (for volume elements local to the charge)

$$\phi(t) \;\propto\; \int \frac{\rho(t - r/c)}{r}\, dV \quad . \tag{5}$$

This has the mathematical character that changes in charge information at the origin propagate out to radial distances r in time $r = ct$, in accord with causality. Now it is straightforward to extend this to distributed charge populations, by just offsetting the spatial origin. For volumes V' populated by a charge density $\rho(\mathbf{r}',\, t)$, the full potential is

$$\phi(\mathbf{r},\, t) \;=\; \int \rho\left(\mathbf{r}',\, t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \right) \frac{d^3x'}{|\mathbf{r} - \mathbf{r}'|} \quad . \tag{6}$$

The vector potential analog is obtained by virtually identical considerations, with the charge density being replaced by the current density. Thus,

$$\mathbf{A}(\mathbf{r},\, t) \;=\; \frac{1}{c}\int \mathbf{j}\left(\mathbf{r}',\, t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \right) \frac{d^3x'}{|\mathbf{r} - \mathbf{r}'|} \quad . \tag{7}$$

These two generalized, causal forms for the components of the 4-potential are referred to as **retarded potentials**. Disturbances in the 4-current have their impact on the field realized a distance ct away.

- In the absence of time dependence for j^μ, these reduce to the familiar electrostatic and magnetostatic forms, respectively.

1.2 The Liénard-Wiechart Retarded Potentials

A key consequence of Maxwell's equations is that accelerating charges radiate electromagnetic radiation. Here the formalism needed to explore such radiation is progressively assembled, centered on the well-known **Larmor formula**. We start by considering potentials in a relativistic construct. **L&L, Sec. 63**

Plot: Retarded Potential Geometry

Consider a charge q moving along a trajectory $\mathbf{r} = \mathbf{r}_0(t)$ with velocity vector $\mathbf{u}(t) = \dot{\mathbf{r}}_0(t)$. We can immediately write down the **four-current density** j^μ to express its charge and current properties: **R&L, Sec. 3.1**

$$j^\mu(x^\mu) \; = \; qc\,\beta^\mu\,\delta\left(\mathbf{r} - \mathbf{r}_0(t)\right) \quad , \quad j^\mu \; \equiv \; (\rho c,\, \mathbf{j}) \quad , \tag{8}$$

for $x^\mu = (ct,\, \mathbf{r})$ and a four-velocity $\beta^\mu = (1,\, \mathbf{u}/c)$. This trivially integrates over volume $d^3\mathbf{r}$ to yield the total charge and current. The scalar and vector potential can be combined as a **four-potential** $A^\mu = (\phi,\, \mathbf{A})$ and expressed is integrals over the four-current

$$A^\mu(x^\mu) \; = \; \frac{1}{c}\int d^3\mathbf{r}' \int dt' \frac{j^\mu(ct',\, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\,\delta\left(t' - t + \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right) \tag{9}$$

at all points in space and time. This is a straightforward adaption of the combination of Eq. (6) and (7) to express an integration over all past times t' as the charge moves along its trajectory. Thus, this defines the cumulative contribution of causally-connected radiation fields generated at the observation point $\mathbf{r}(t)$ and time t.

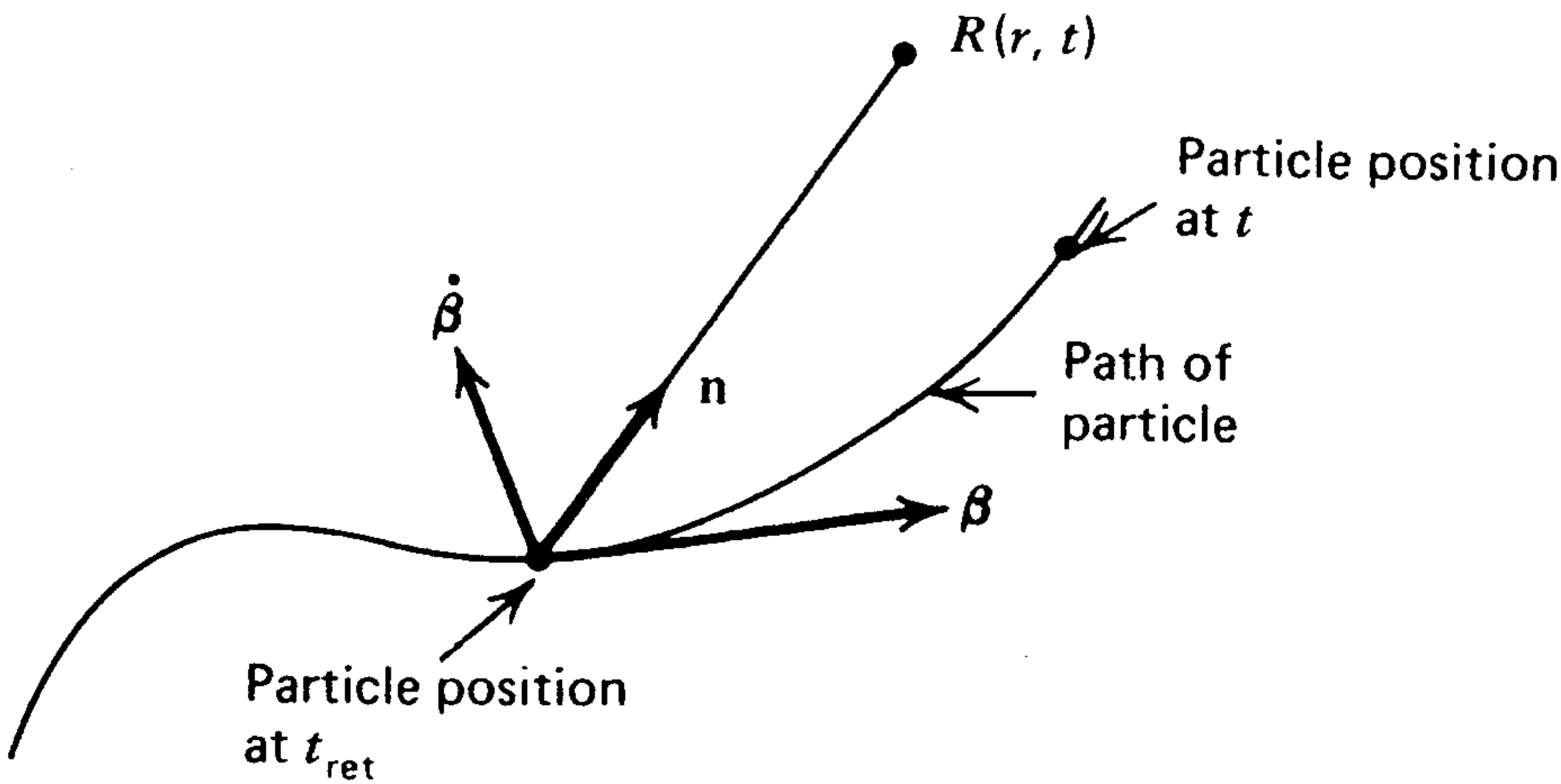
- For our single charge, the space integral is trivial, replacing $\mathbf{r}'$ by $\mathbf{r}_0(t')$. It is this motion of the charge that renders the t' integration more involved.

To evaluate the δ function, we introduce a change of variables that simplifies its argument. To this end, we define

$$\mathbf{R}(t') \; = \mathbf{r} - \mathbf{r}_0(t') \quad , \quad R(t') \; = \; \left|\mathbf{R}(t')\right| \quad , \quad \mathbf{n} \; = \; \frac{\mathbf{R}}{R} \quad . \tag{10}$$

Here $\mathbf{R}(t')$ is the *look-back* position vector, possessing a unit vector $\mathbf{n}$, and $t' \; = \; t - \mathbf{R}(t')/c$ the *look-back* or **retarded time** to past epochs of the

Retarded Potential Geometry



- The world line geometry of a moving charge used for calculating its radiation field at distance R in direction **n** using retarded potentials.
- Fig. 3.1 of Rybicki & Lightman: *Radiative Processes in Astrophysics* (1979).

particle's motion that contribute to the potentials at the present x^μ. The solution of the δ function defines the retarded time $t_{\rm ret}$:

$$\eta\,(t_{\rm ret}) \;\equiv\; c(t_{\rm ret} - t) + R\,(t_{\rm ret}) \;=\; 0 \quad . \tag{11}$$

This forges *causal connection* to all past history of the particle along its trajectory. By changing variables to $t'' = c\,\eta(t')$, the evaluation of the δ function introduces a factor of $c/\eta'(t_{\rm ret})$, with the derivative being given by

$$\eta'(t_{\rm ret}) \;=\; c + \frac{\partial R}{\partial t'}\bigg|_{t'=t_{\rm ret}} \;=\; c - \mathbf{n}\cdot\mathbf{u}(t_{\rm ret}) \quad . \tag{12}$$

At this point we introduce an **aberration factor**

$$\kappa(t_{\rm ret}) \;=\; 1 - \mathbf{n}(t_{\rm ret}).\boldsymbol{\beta}(t_{\rm ret}) \quad , \quad \boldsymbol{\beta} \;=\; \frac{\mathbf{u}}{c} \quad . \tag{13}$$

The final evaluation of the time integral in Eq. (9) yields

$$A^\mu(x^\mu) \;=\; \left[\frac{q\,\beta^\mu}{\kappa\,R}\right]_{\rm ret} \quad , \quad \beta^\mu \;=\; \frac{u^\mu}{\gamma} \;=\; \left(1,\, \frac{\mathbf{u}}{c}\right) \quad . \tag{14}$$

These are the **Liénard-Wiechart retarded potentials**, with the subscript *ret* denoting evaluation of all quantities at the retarded time that is a solution of Eq. (11), to accommodate relativistic causality.

• The aberration factor κ dictates a concentration of the potentials in a narrow cone about the particle velocity, what is called a **relativistic beaming** effect, which is coupled to **Doppler boosting**.

• The appearance of the retarded time is of central importance: *it makes it possible for the particle to radiate*: instead of the fields possessing $1/R^2$ dependence, evaluation of potentials at $t_{\rm ret}$ drives a $1/R$ dependence to the radiation field. This will become evident shortly.

1.3 Fourier Representation of Retarded Potentials

The definition of the Fourier representation of the potentials and fields of moving charges, including the inherent retardation, can be derived by adaptation of the corresponding analysis of the electrostatic Coulomb field in Chapter 8. Here we have a wave equation as the starting point: **L&L, Sec. 64**

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \;=\; -4\pi q\,\delta^3(\mathbf{r}-\mathbf{v}\,t) \;\equiv\; -4\pi q\int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}\,t)}\,\frac{d^3k}{(2\pi)^3} \quad , \tag{15}$$

where the charge distribution is $\rho = q\,\delta^3(\mathbf{r}-\mathbf{v}\,t)$ for a charge q moving with velocity $\mathbf{v}$. Again the delta function Fourier representation identity has been invoked on the right. Hereafter, we will employ $\mathbf{v} = \boldsymbol{\beta}c$. Writing

$$\phi \;=\; \int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{r}}\,\phi_{\mathbf{k}}\,\frac{d^3k}{(2\pi)^3} \quad\Rightarrow\quad \nabla^2\phi \;=\; \int_{-\infty}^{\infty}(ik)^2\,e^{i\mathbf{k}\cdot\mathbf{r}}\,\phi_{\mathbf{k}}\,\frac{d^3k}{(2\pi)^3} \tag{16}$$

leads to the identification of an inhomogeneous ODE in the time variable for $\phi_{\mathbf{k}}$ and its routine solution:

$$\frac{1}{c^2}\frac{\partial^2\phi_{\mathbf{k}}}{\partial t^2} + k^2\,\phi_{\mathbf{k}} \;=\; -4\pi q\,e^{-i\mathbf{k}\cdot\boldsymbol{\beta}\,ct} \quad\Rightarrow\quad \phi_{\mathbf{k}} \;=\; 4\pi q\,\frac{e^{-i\mathbf{k}\cdot\boldsymbol{\beta}\,ct}}{k^2-(\mathbf{k}\cdot\boldsymbol{\beta})^2} \quad . \tag{17}$$

This obviously reduces to the familiar Coulomb form when $\mathbf{v}\to\mathbf{0}$. One can then quickly identify that for a wavevector $\mathbf{k}$, the wave frequency of a fluctuating field is just $\omega = \mathbf{k}\cdot\mathbf{v}$. Derivation of the Fourier amplitude for the vector potential is similar, replacing $\phi\to\mathbf{A}$ and $\rho\to\mathbf{j}/c = q\,\boldsymbol{\beta}\,\delta^3(\mathbf{r}-\mathbf{v}\,t)$, resulting in

$$\mathbf{A}_{\mathbf{k}} \;=\; 4\pi q\,\frac{\boldsymbol{\beta}\,e^{-i\mathbf{k}\cdot\boldsymbol{\beta}\,ct}}{k^2-(\mathbf{k}\cdot\boldsymbol{\beta})^2} \quad . \tag{18}$$

The electric and magnetic field Fourier amplitudes are routinely obtained:

$$\begin{aligned}
\mathbf{E}_{\mathbf{k}} \;&=\; -i\mathbf{k}\,\phi_{\mathbf{k}} + i(\mathbf{k}\cdot\boldsymbol{\beta})\,\mathbf{A}_{\mathbf{k}} \;=\; 4\pi i\,q\,\frac{(\mathbf{k}\cdot\boldsymbol{\beta})\,\boldsymbol{\beta}-\mathbf{k}}{k^2-(\mathbf{k}\cdot\boldsymbol{\beta})^2}\,e^{-i\mathbf{k}\cdot\boldsymbol{\beta}\,ct} \quad , \\
\mathbf{B}_{\mathbf{k}} \;&=\; -i\mathbf{k}\times\mathbf{A}_{\mathbf{k}} \;=\; 4\pi i\,q\,\frac{\mathbf{k}\times\boldsymbol{\beta}}{k^2-(\mathbf{k}\cdot\boldsymbol{\beta})^2}\,e^{-i\mathbf{k}\cdot\boldsymbol{\beta}\,ct} \quad .
\end{aligned} \tag{19}$$

These are derived without reference to whether $\boldsymbol{\beta}$ is constant or not.

1.4 Electromagnetic Fields from Accelerating Charges

The calculation of the fields resulting from the Liénard-Wiechart potentials in Eq. (14) is a routine but lengthy exercise in vector calculus: refer to Jackson (1975), Section 14.1. The electric field can be divided into two contributions, a **velocity field** $\mathbf{E}_{\rm vel}$ and a **radiation field** $\mathbf{E}_{\rm rad}$ (or acceleration field): **R & L, Sec. 3.2**

$$\begin{aligned} \mathbf{E}_{\rm vel} &= q \left[\frac{(\mathbf{n}-\boldsymbol{\beta})\,(1-\beta^2)}{\kappa^3 R^2} \right]_{\rm ret} \quad , \\ \mathbf{E}_{\rm rad} &= \frac{q}{c} \left[\frac{\mathbf{n}}{\kappa^3 R} \times \left\{ (\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right\} \right]_{\rm ret} \quad . \end{aligned} \tag{20}$$

The total electric and magnetic fields spawned by the moving charge are

$$\mathbf{E}(\mathbf{r},\, t) \;=\; \mathbf{E}_{\rm vel} + \mathbf{E}_{\rm rad} \quad , \quad \mathbf{B}(\mathbf{r},\, t) \;=\; \Big[\mathbf{n} \times \mathbf{E}(\mathbf{r},\, t) \Big]_{\rm ret} \quad . \tag{21}$$

The relativistic influences inherent in these formulae include:

- an aberration factor $(\mathbf{n}-\boldsymbol{\beta})/\kappa$, present in both fields;
- length contraction factors $\sqrt{1-\beta^2}/\kappa$, two in $\mathbf{E}_{\rm vel}$ and one in $\mathbf{E}_{\rm rad}$;
- and a time dilation factor γ/κ in $\mathbf{E}_{\rm rad}$ due to the time derivative.

These appear due to the time and space differentiation operations.

Plot: Geometry from Fig. 3.1 of R&L.

• The velocity field $\mathbf{E}_{\rm vel}$ is the only contribution in the absence of acceleration. It represents a transient electromagnetic pulse as the charge whisks by, but since

$$\int |\mathbf{E}_{\rm vel}|^2\, R^2\, d\Omega \;\to\; 0 \quad \text{as} \quad R \;\to\; \infty \quad , \tag{22}$$

it conveys no electromagnetic energy to infinity. It is a relativistic generalization of Coulomb's law, collapsing to the familiar result when $\beta \ll 1$.

• The second term, the radiation field, is the result of the acceleration of the charge. Since it falls off as $1/R$, it transmits a finite value of electromagnetic energy to infinity, as radiation propagated in the direction $\mathbf{n}$ with fields $\mathbf{E}_{\rm rad}$ and $\mathbf{B}_{\rm rad} = \mathbf{n} \times \mathbf{E}_{\rm rad}$, *both perpendicular to* $\mathbf{n}$.

• As a *gedanken experiment*, consider a charge in linear motion rapidly decelerating on a timescale Δt centered on time $t = 0$ and stopping point $x = 0$. The fields are schematically illustrated in Fig. 3.2 of R&L.

$\boxed{\textbf{Plot:}}$ Electric Field of a Decelerating Charge

∗ The **near-field zone**, $R \lesssim ct$, (for $t > 0$) is described by the dominant $1/R^2$ term from the *stopped charge*, i.e. a Coulomb field $\mathbf{E}_{\text{vel}}$ for the specific case of $\beta = 0$.

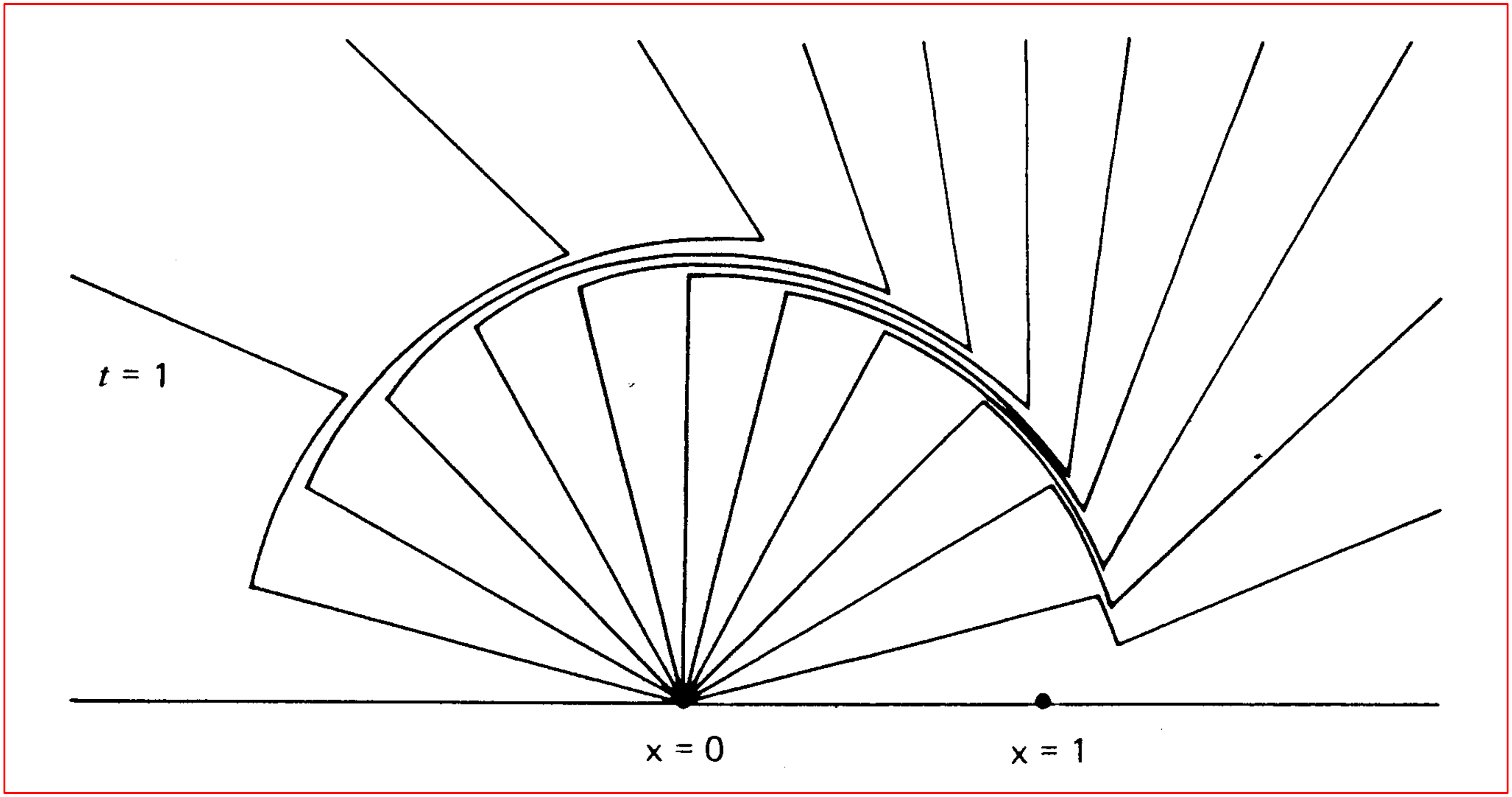
∗ The **far-field zone**, $R \gtrsim ct$, is also a Coulomb field, but corresponding to the velocity field $\mathbf{E}_{\text{vel}}$ for the $\beta > 0$ charge prior to deceleration. It is centered on the advance point $x = ct$, and possesses the characteristic disk-like compression of the field lines perpendicular to $\boldsymbol{\beta}$.

• The discontinuity or interface separating these two regions travels at speed c, and represents an electromagnetic wave. The thickness of this interface is $c\Delta t$. Its cross section is an annular ring of radius $R \sim ct$, threaded by bunched field lines that yield total conservation of electric flux.

∗ Accordingly, the field within this interface must scale as $|\mathbf{E}_{\text{rad}}| \sim 1/R$, so that E/M energy is thereby transported to infinity as Poynting flux. In reality, this energy is eventually absorbed by other charges in its path.

∗ This idealized, central charge case obviously can be adapted to charge ensembles, for which centrality goes away, and the radiation wave is no longer isotropic. Such is true also for gradual acceleration/deceleration of a single charge, since there are two special directions provided by $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$.

Electric Field of a Decelerating Charge



- Schematic of the 1/R radiation field of a decelerating charge. The charge impulsively decelerates from constant velocity v to zero speed (i.e. becomes stationary) at time t=0 at the origin.
- Fig. 3.2 of Rybicki & Lightman: *Radiative Processes in Astrophysics* (1979).