Eliminating the explicit time dependence gives

$$\frac{E_y^2}{E_1^2} + \frac{E_z^2}{E_2^2} \;=\; 1 \quad . \tag{31}$$

Therefore, as time progresses, the electric vector rotates in the plane orthogonal to the direction of propagation, with its tip tracing out an ellipse in the (y, z)-plane. This is the most general form of a monochromatic wave, and it is said to be **elliptically polarized**.

Plot: The Polarization Ellipse

• If either of E_1 or E_2 is zero, then the wave is described by just a single sinusoid, and is said to be **linearly polarized**. This is the "purest" polarization configuration for electromagnetic waves, and can be generated by an alternating current in a wire: the electric field vector is confined to a plane that contains the **k** vector also.

• In the special case that $E_1 = E_2 \neq 0$, then the ellipse reduces to a circle and the wave is said to be **circularly polarized**. These are a superposition of two linearly polarized waves of equal **electric amplitude** but orthogonal polarization (i.e., **E**-field vector direction), and a single phase offset.

$*$ *Elliptically polarized* waves are a superposition of two linearly polarized waves of unequal amplitude, orthogonal polarization, and a phase offset.

Now define a four-dimensional wavevector that satisfies a massless condition:

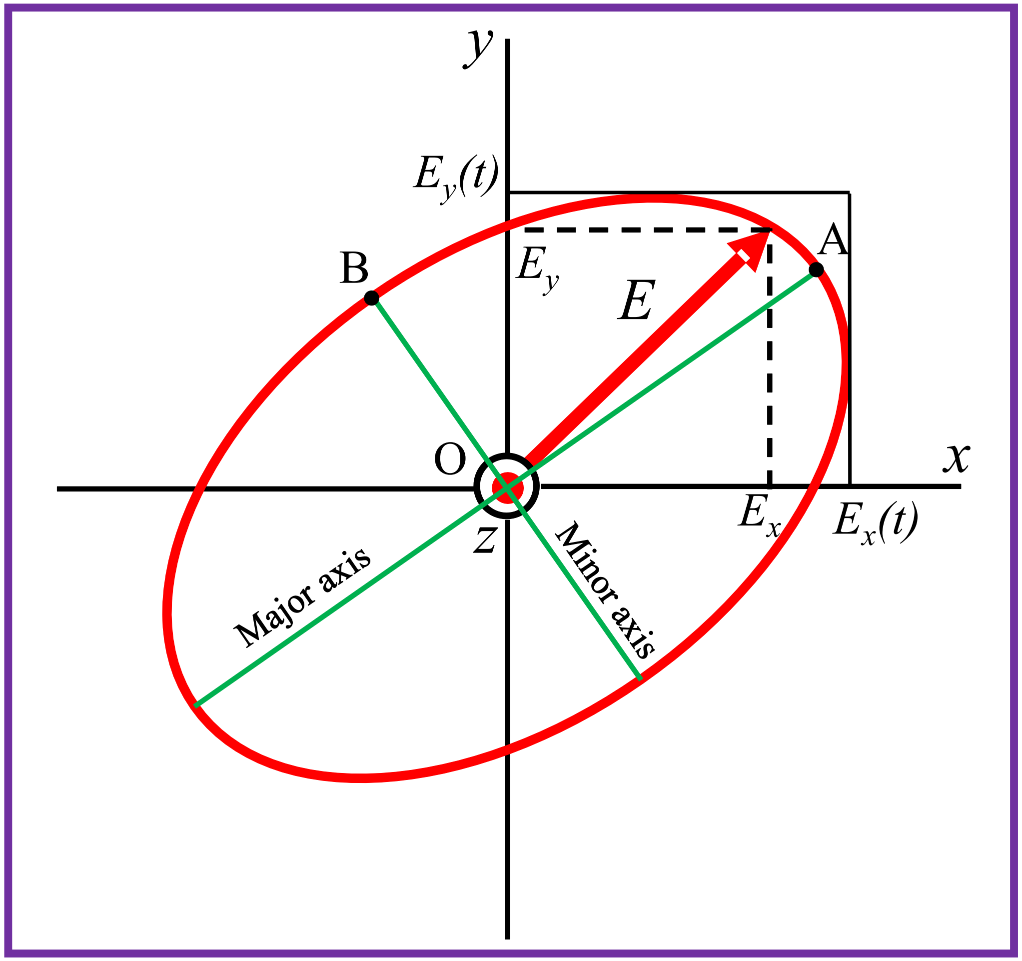
$$k^\alpha \;=\; \left(\frac{\omega}{c}, \mathbf{k}\right) \quad \Rightarrow \quad k^\alpha k_\alpha \;=\; 0 \quad . \tag{32}$$

The argument of the complex exponential in the Fourier transform for the vector potential is the negative of

$$k_\alpha x^\alpha \;=\; \omega t - \mathbf{k}\cdot\mathbf{r} \quad . \tag{33}$$

This is necessarily a Lorentz invariant (scalar), and so it follows that k^α must be a true 4-vector. Accordingly, the Lorentz transformation properties

The Polarization Ellipse



- The elliptical path in the (x,y) plane of the electric field vector ***E*** for an electromagnetic wave propagating in the z-direction.
- From Fig. 2.14 of A. Pal, PhD Thesis, Swansea Univ. (2013).

are easily specified. Because of the massless condition relating space and time components, we need only focus on the time component mapping:

$$k_0 \;=\; \gamma\bigl(k_0' - \beta k_1'\bigr) \quad . \tag{34}$$

If the direction between the propagation vector in the K' frame and the boost $\boldsymbol{\beta}$ is θ', then since $k_1'/k_0' \to \cos\theta'$, the perceived frequency of the wave in the K frame is

$$\omega_0 \;=\; \gamma(1 - \beta\cos\theta')\,\omega' \quad . \tag{35}$$

This change in frequency is known as the **Doppler effect** for light, and is an increase (decrease) according to the K' frame approaching (receding).

2 Polarization Formalism: Stokes Parameters

The preceding examples explored *fully-polarized light*, where there is a coherent correlation between two linear polarization states superposed to produce an elliptically polarized wave. In general, such a perfect configuration often does not exist, and the mixes of light waves are somewhat incoherent, which means they are not monochromatic, and possess a spread of frequencies ω, wavenumbers $\mathbf{k}$ and/or phases ϕ. We therefore consider an electric field vector function **L&L Sec. 50**

$$\mathbf{E} \;=\; \mathbf{E}_0(t)\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \;\approx\; (\mathbf{E}_1 + i\mathbf{E}_2)\exp\Bigl\{ i(\mathbf{k}\cdot\mathbf{r} - [\omega + \Delta\omega]\, t - \alpha)\Bigr\} \quad , \tag{36}$$

and suppress the explicit display of $\Delta\mathbf{k}$ and $\Delta\phi$ information. Experimentally, we do not measure the electric field of light waves, just the power it transmits into a "detector" or medium. Therefore, *we are interested in quadratic quantities involving the electric field amplitude.* The magnetic field is not specifically considered in the subsequent presentation, though it is easily obtained.

In what follows, it is presumed that $\mathbf{k} \propto \hat{x}$, i.e. the wave propagates in the x-direction, and the polarization information is contained in the (y, z)-plane.

2.1 The Polarization 2-Tensor

For the remainder of this Section, we will concern ourselves with the time portion of the electric field profile. Measurements cannot resolve the time variations, so one needs only to consider time averages $\langle\rangle_t$ on scales much greater than $2\pi/\omega$. In forming quadratic scalars involving $\mathbf{E(t)}$, combinations not involving complex conjugates, i.e., for $i, j = 1, 2$,

$$E_i E_j \;\propto\; E_{0i} E_{0j} e^{-2i\omega t} \quad , \quad E_i^* E_j^* \;\propto\; E_{0i}^* E_{0j}^* e^{2i\omega t} \quad , \tag{37}$$

average to zero over long times. Therefore we are only interested in products that mix coefficients of the spatial portion of the field and complex conjugates. We define the **intensity tensor** for a light wave, a 2-tensor, as

$$I_{ij} \;=\; c \left\langle \mathcal{E}_i \mathcal{E}_j^* \right\rangle_t \tag{38}$$

for

$$\mathbf{E}(\mathbf{r},\, t) \;\equiv\; \mathbf{E}_0(t)\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \;=\; \Big(\mathcal{E}_1 \hat{y} + \mathcal{E}_2 \hat{z}\Big) e^{-i\omega t} \quad . \tag{39}$$

Thus, the $\mathcal{E}_i(\mathbf{r}, t)$ are coefficients that contain the space information of the wave, the slowly varying time information, and the electric field vector information. With the time-averaging process, the spatial dependence factors out. The trace of this 2-tensor is

$$I \;\equiv\; I_{jj} \;=\; c \left\langle \mathcal{E}_1 \mathcal{E}_1^* + \mathcal{E}_2 \mathcal{E}_2^* \right\rangle_t \;=\; c \left\langle \mathbf{E}_0(t)\, \mathbf{E}_0^*(t) \right\rangle_t \quad , \tag{40}$$

and is termed the **intensity** of the light wave. It has units of energy per unit area per unit time (contrasting the L&L definition of energy/volume).

The intensity is the measure of how much light we have. With a polaroid detector, we are interested in retaining the polarization information. The intensity tensor is then scaled by its trace, to form the **polarization tensor**:

$$P_{ij} \;=\; \frac{I_{ij}}{I} \;=\; \frac{\left\langle \mathcal{E}_i \mathcal{E}_j^* \right\rangle_t}{\left\langle \mathcal{E}_1 \mathcal{E}_1^* + \mathcal{E}_2 \mathcal{E}_2^* \right\rangle_t} \quad \text{with} \quad P_{ij} \;=\; P_{ji}^* \quad . \tag{41}$$

It is a dimensionless 2-tensor that is obviously **Hermitian**, and *has a trace of unity*. Consequently, the diagonal components are real and sum to unity.

• For completely linearly-polarized light, clearly $\mathbf{E}_0(t)$ is independent of time, and so the time-averaging process is immaterial. One can then write

$$I_{ij} \; = \; I\, P_{ij} \; = \; \mathcal{E}_i \mathcal{E}_j \quad , \tag{42}$$

i.e. a product of two constant vectors. Hence, as P_{ij} is diagonalizable as a matrix, i.e., can be brought into diagonal form via a rotation in the $(y,\, z)$ plane, in this special case it must have zero determinant (one $\mathcal{E}_i$ is zero):

$$\big| P_{ij} \big| \; = \; P_{11} P_{22} - P_{12} P_{21} \; \equiv \; P_{yy} P_{zz} - P_{yz} P_{yz}^* \; = \; 0 \quad . \tag{43}$$

Here the correspondence $1 \to y\,,\; 2 \to z$ is highlighted. The tensor components are therefore restricted by this additional equation.

For natural or completely unpolarized light, all directions in the (y, z)-plane are equivalent. The polarization tensor must then be diagonal so that

$$P_{ij} \; = \; \frac{\delta_{ij}}{2} \quad \Rightarrow \quad \big| P_{ij} \big| \; = \; \frac{1}{4} \quad . \tag{44}$$

These two highlight cases establish the extremes for the physical range of allowable determinants for the polarization tensor: $0 \leq |P_{ij}| \leq 1/4$. This leads to the definition of the **total degree of polarization** P via

$$P \; = \; \sqrt{1 - 4 \big| P_{ij} \big|} \quad , \quad 0 \; \leq \; P \; \leq \; 1 \quad . \tag{45}$$

This is often also quoted in percentages.

For more general circumstances, the polarization tensor can be conveniently deconstructed into essential elements by isolating its symmetric and anti-symmetric portions. For a Hermitian tensor this is simply done:

$$P_{ij} \;\; = \;\; \underbrace{\frac{1}{2}\big(P_{ij} + P_{ji} \big)}_{S_{ij}} + \underbrace{\frac{1}{2}\big(P_{ij} - P_{ji} \big)}_{-\frac{i}{2} \epsilon_{ij}\, A} \tag{46}$$

This isolates the *symmetric part* S_{ij}, which is purely real, and the anti-symmetric part, which is proportional to real A and is imaginary. Here

$$\epsilon_{ij} \; = \; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{47}$$

is the unit *anti-symmetric* tensor of rank 2.

• For *unpolarized light*, since P_{ij} is diagonal, then $A = 0$. For *linearly-polarized light*, while P_{ij} is not necessarily diagonal, one has the freedom to choose $\mathbf{E}_0$ to be real, for which $A = 0$ also. Then, again P_{ij} is real and diagonal. For *circularly-polarized light*, the magnitudes of $\mathbf{E}_1$ and $\mathbf{E}_2$ have to be identical, so that $\mathcal{E}_2 = i\mathcal{E}_1$. It then follows that

$$S_{ij} \;=\; \delta_{ij} \quad \text{and} \quad A \;=\; \pm 1 \quad \Rightarrow \quad P_{ij} \;=\; \frac{\delta_{ij}}{2} \mp \frac{i}{2}\,\epsilon_{ij} \quad . \tag{48}$$

The value of A is referred to as the **helicity** of the wave, and it is **right-hand polarized** for $A = 1$ an **left-hand polarized** for $A = -1$.

The elliptical polarization character of a wave will, in general, establish elliptic axes for the electric field vector evolution that do not coincide with the coordinate axis an observer or measurement will adopt. Accordingly, a suitable transformation is needed. Let the wave's elliptical axes be along unit vectors $\mathbf{n}_y$ and $\mathbf{n}_z$ that are orthogonal to each other, i.e. $\mathbf{n}_y \cdot \mathbf{n}_z = 0$. *These define the directions of the electric field vectors of the two constituent linear polarization states that superpose to generate the complete wave.* They can be terms the **principal axes** of the elliptical polarization state.

The polarization tensors for each of these linear polarizations have $A = 0$ and possess just the symmetric part:

$$S_{ij} \;=\; n_{y,i}\, n_{y,j} \quad \text{or} \quad S_{ij} \;=\; n_{z,i}\, n_{z,j} \quad . \tag{49}$$

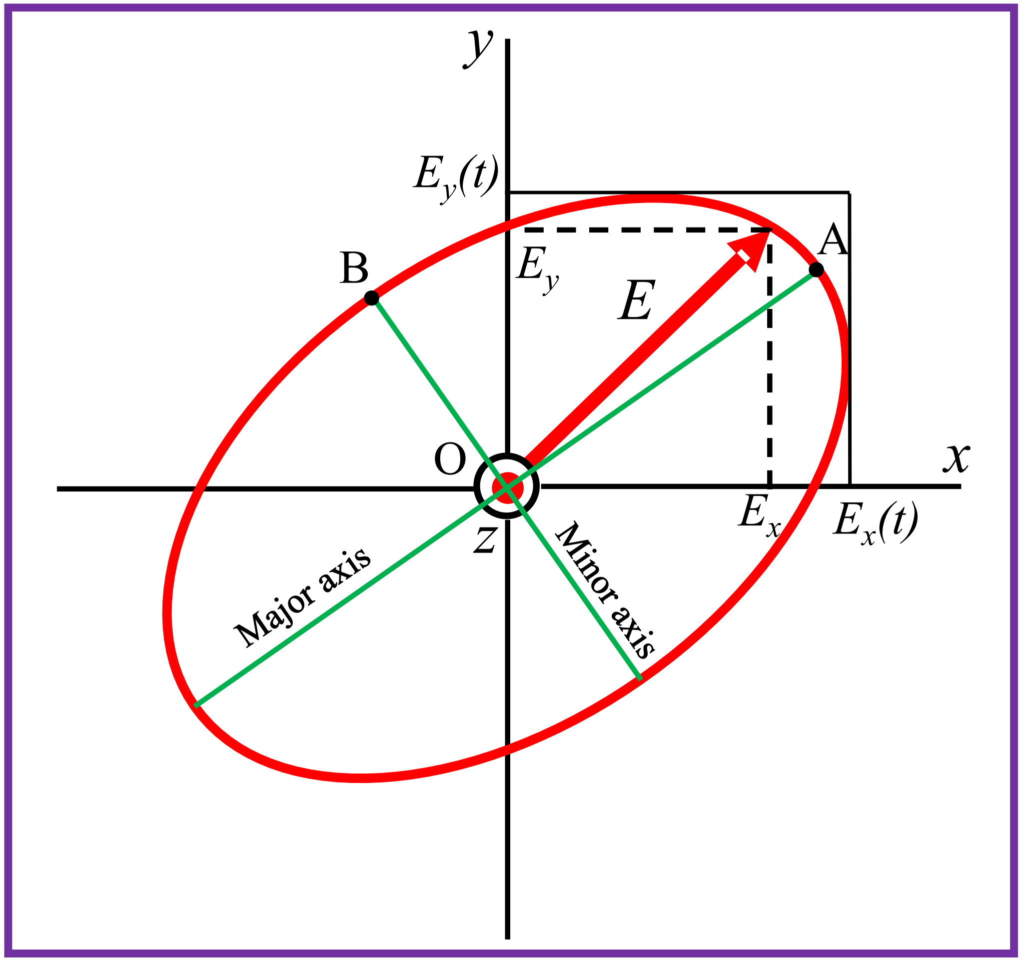
These each have only one element that is non-zero and are clearly diagonal tensors with zero determinant, corresponding to $P = 1$ degree of polarization. The linear superposition of these for the purely elliptical polarization state therefore has a diagonal symmetric tensor component with

$$S_{ij} \;=\; \lambda_y\, n_{y,i}\, n_{y,j} + \lambda_z\, n_{z,i}\, n_{z,j} \;\to\; \begin{pmatrix} \lambda_y & 0 \\ 0 & \lambda_z \end{pmatrix} \quad , \quad \lambda_y + \lambda_z \;=\; 1 \quad . \tag{50}$$

The determinant of S_{ij} is just $\lambda_y \lambda_z$, and this can be expressed in terms of $P_{\rm lin}$, the degree of polarization from a superposition of purely linear polarization modes, as a opposed to a mix including circular polarization. Thus the determinant is $\lambda_y \lambda_z = (1 - P_{\rm lin}^2)/4$, and can be solved for λ_y and λ_z as

$$\lambda_y \;=\; \frac{1 + P_{\rm lin}}{2} \quad , \quad \lambda_z \;=\; \frac{1 - P_{\rm lin}}{2} \quad . \tag{51}$$

The Polarization Ellipse



- The elliptical path in the (x,y) plane of the electric field vector ***E*** for an electromagnetic wave propagating in the z-direction.
- From Fig. 2.14 of A. Pal, PhD Thesis, Swansea Univ. (2013).

One has the freedom to rotate the principal axes to render $\lambda_y > \lambda_z$ (y is the semi-major axis of the polarization ellipse). Then

$$S_{ij} \; = \; S_{\rm P} \; \equiv \; \frac{1}{2} \begin{pmatrix} 1 + P_{\rm lin} & 0 \\ 0 & 1 - P_{\rm lin} \end{pmatrix} \tag{52}$$

is the form for the symmetric part of the polarization tensor when the coordinate axes are aligned with the principal axes of the ellipse. General coordinate axes are not so aligned, and can be obtained via a simple rotation through **polarization angle** χ (L&L use ϕ) in the y, z plane. Thus

$$\begin{aligned} S_{ij} \; &= \; \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix}^T S_{\rm P} \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \\ &= \; \frac{1}{2} \begin{pmatrix} 1 + P_{\rm lin}\cos 2\chi & P_{\rm lin}\sin 2\chi \\ P_{\rm lin}\sin 2\chi & 1 - P_{\rm lin}\cos 2\chi \end{pmatrix} \quad . \end{aligned} \tag{53}$$

This is the general form of the symmetric portion of the polarization tensor, and is specified in terms of two **radiation parameters**: $P_{\rm lin}$ and χ.

Plot: The Polarization Ellipse Revisited

Note the rotational period of π in the χ variable. When we add the anti-symmetric portion, we introduce a third parameter, A, the degree of circular polarization. This completes the specification of a non-monochromatic wave.

2.2 Stokes Parameters

These three radiation parameters are commonly expressed via the following well-known choice:

$$Q \; = \; I\, P_{\rm lin} \cos 2\chi \quad , \quad U \; = \; I\, P_{\rm lin} \sin 2\chi \quad , \quad V \; = \; I\, A \quad . \tag{54}$$

These are known as **Stokes parameters**[1]; Q and U specify the linear polarization magnitude and orientation, and V expresses the circularity of the polarization. Clearly Q/I, U/I and V/I all range between -1 and 1.

[1]Obtained by George Gabriel Stokes in 1852.

The distinct advantage of them is that they are additive for radiation from different origins or zones of interest [*Draw*]. The polarization tensor becomes

$$P_{ij} \;=\; \frac{1}{2I}\begin{pmatrix} I+Q & U-iV \\ U+iV & I-Q \end{pmatrix} \quad , \tag{55}$$

and its determinant is

$$\bigl|P_{ij}\bigr| \;=\; \frac{1}{4I^2}\Bigl(I^2-Q^2-U^2-V^2\Bigr) \;=\; \frac{1}{4}\bigl(1-P^2\bigr) \quad . \tag{56}$$

From this one simply determines that the **total polarization degree** is

$$P \;=\; \frac{\sqrt{Q^2+U^2+V^2}}{I} \quad . \tag{57}$$

Therefore, P is clearly not additive when summing over different regions of interest or radiation fields. Furthermore, we discern that all three Stokes parameters are bounded by IP, so that one can also define a second polarization angle ψ that is often convenient: $A = P\sin 2\psi$. Substitution into Eq. (57) then obtains

$$P_{\rm lin} \;\equiv\; \frac{\sqrt{Q^2+U^2}}{I} \;=\; P\cos 2\psi \quad . \tag{58}$$

It therefore follows that the three Stokes parameters can be specified in terms of the polarization degree P and angles χ and ψ:

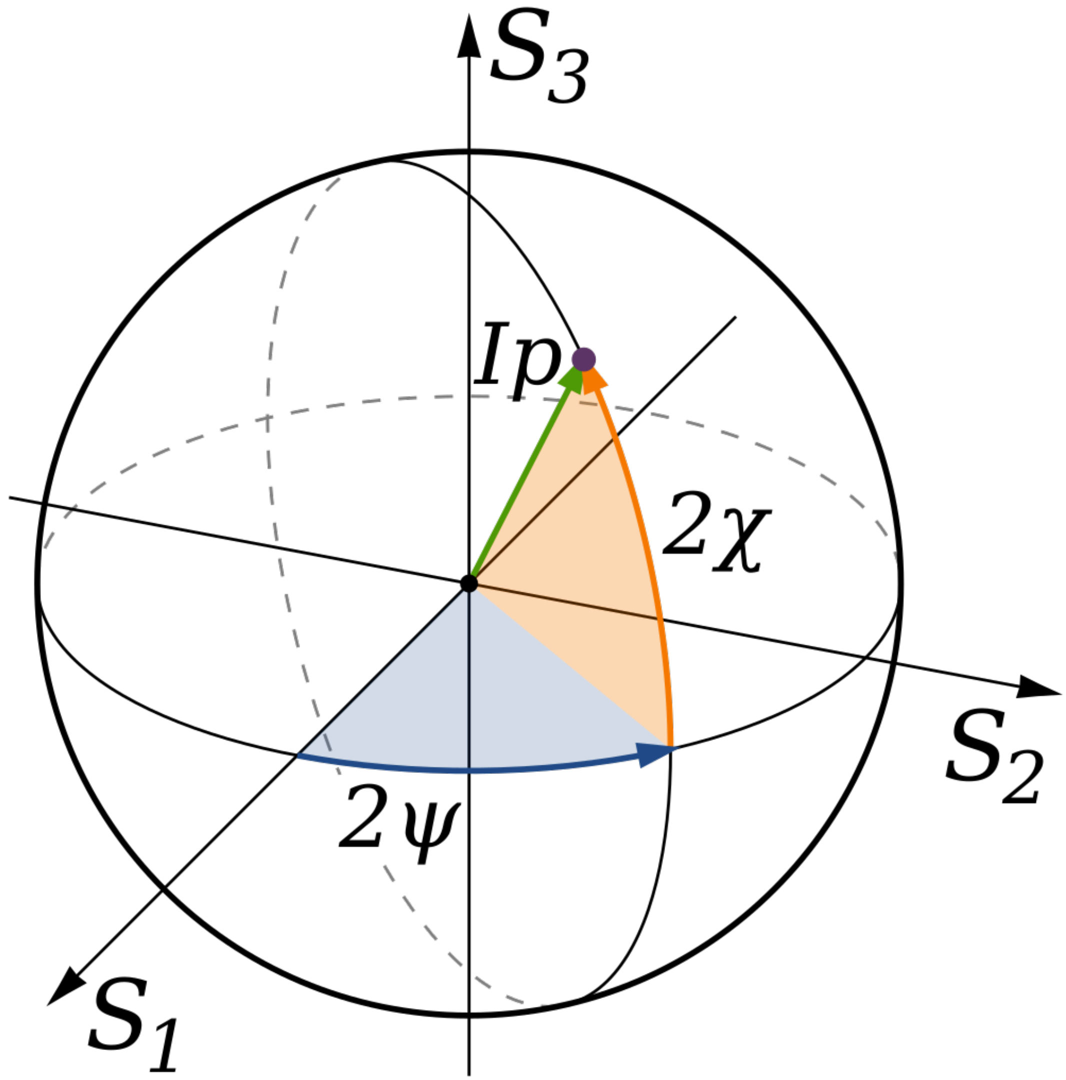
$$\frac{Q}{I} \;=\; P\,\cos 2\chi\,\cos 2\psi \;, \quad \frac{U}{I} \;=\; P\,\sin 2\chi\,\cos 2\psi \;, \quad \frac{V}{I} \;=\; P\sin 2\psi \;. \tag{59}$$

The polarization state can be represented on a 2D surface construction known as the **Poincaré sphere**, where χ, ψ define spherical polar coordinates.

Plot: Polarization: the Poincaré sphere

- Note that while these developments leveraged the elliptical polarization state, they can apply to any non-monochromatic radiation configuration.

Polarization: Poincaré Sphere



- Diagram from Wikimedia Commons.