9. ELECTROMAGNETIC WAVES

Matthew Baring — Lecture Notes for PHYS 532, Spring 2023

1 The Electromagnetic Wave Equation

In a localized vacuum, the four-current is $j^\mu = 0$, yet Maxwell's equations admit a solution. They take the form **L&L Sec. 46**

$$\begin{aligned} \nabla\cdot\mathbf{E} \;&=\; 0 \quad , \quad \nabla\times\mathbf{E} \;=\; -\frac{1}{c}\,\frac{\partial\mathbf{B}}{\partial t} \quad \text{[Faraday]} \quad , \\ \nabla\cdot\mathbf{B} \;&=\; 0 \quad , \quad \nabla\times\mathbf{B} \;=\; \frac{1}{c}\,\frac{\partial\mathbf{E}}{\partial t} \quad \text{[Ampere]} \quad . \end{aligned} \tag{1}$$

If the fields are presumed time-independent, then our studies of electrostatics and magnetostatics automatically imply the trivial solution $\mathbf{E} = \mathbf{0} = \mathbf{B}$. Accordingly, non-trivial solutions must be time-dependent, and in fact exist.

For convenience, the developments are made using the vector potential. Using its definition and inserting it into Faraday's law for the $\partial\mathbf{B}/\partial t$ term,

$$\mathbf{B} \;=\; \nabla\times\mathbf{A} \quad\Rightarrow\quad \mathbf{E} \;=\; -\frac{1}{c}\,\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi \quad . \tag{2}$$

The scalar potential is eliminated in L&L immediately through a gauge choice, but doesn't really need to be yet. These two equations can be inserted into Ampere's law to yield a second-order PDE for the vector and scalar potentials:

$$\nabla\times\bigl(\nabla\times\mathbf{A}\bigr) \;\equiv\; \nabla\bigl(\nabla\cdot\mathbf{A}\bigr) - \nabla^2\mathbf{A} \;=\; -\frac{1}{c^2}\,\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\Bigl(\frac{1}{c}\,\frac{\partial\phi}{\partial t}\Bigr) \quad . \tag{3}$$

This rearranges to

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} \;=\; \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) \quad . \tag{4}$$

L&L choose the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$ together with the trivial potential $\phi = 0$ to render the RHS equal to zero. However it suffices to adopt the <u>covariant</u> **Lorenz gauge** condition

$$\frac{\partial A^\mu}{\partial x^\mu} \;=\; \frac{1}{c}\frac{\partial\phi}{\partial t} + \nabla\cdot\mathbf{A} \;=\; 0 \tag{5}$$

The result is **d'Alembert's wave equation** for the vector potential:

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} \;=\; \mathbf{0} \quad . \tag{6}$$

In 3D this can admit plane wave solutions.

- The wave equation can also admit non-planar solutions. For example, a dipole oscillating in a linear direction, or spinning/precessing in an electric field creates an anisotropic 3D wave that asymptotically at large distances assumes the shape of a spherical front.

- The wave equation can also be derived quickly via covariant formalism. The second pair of Maxwell field equations can be expressed using the covariant form of the electromagnetic field tensor:

$$0 \;=\; \frac{\partial F^{\mu\nu}}{\partial x^\nu} \;=\; \frac{\partial}{\partial x^\nu}\left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}\right) \;=\; \cancel{\frac{\partial}{\partial x_\mu}\frac{\partial A^\nu}{\partial x^\nu}} - \frac{\partial^2 A^\mu}{\partial x^\nu \partial x_\nu} \quad . \tag{7}$$

The cancelled term is chosen to be zero via the Lorenz gauge choice in Eq. (5), which is manifestly covariant. The result is simply the covariant form of the wave equations, four in all:

$$\frac{\partial^2 A^\mu}{\partial x^\nu \partial x_\nu} \;=\; 0 \quad . \tag{8}$$

The potentials are not uniquely determined. If we perform a further gauge transformation $\mathbf{A} \to \mathbf{A} + \nabla f$, $\phi \to \phi - 1/c\,\partial f/\partial t$, then f has to obey the wave equation, but then the new four-potential is a viable solution. One can choose $\partial f/\partial t = c\phi$ to eliminate the scalar potential.

1.1 Plane Waves

The planar solution of Eq. (6) is of particular interest, since it is our standard elemental description for light. d'Alembert's PDE is then routinely solved. This case corresponds to each component of the vector potential depending only on one coordinate direction, which we presume to be x. Then **L&L Sec. 47**

$$\frac{\partial^2 A^i}{\partial t^2} - c^2 \frac{\partial^2 A^i}{\partial x^2} \; = \; 0 \quad . \tag{9}$$

This "factorizes" as

$$\left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) A^i \; = \; 0 \quad , \tag{10}$$

suggest the transformation to **normal coordinates** $\xi = t - x/c$ and $\eta = t + x/c$ that define the characteristics of the PDE. Then

$$\frac{\partial}{\partial \xi} \; = \; \frac{1}{2} \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \quad , \quad \frac{\partial}{\partial \eta} \; = \; \frac{1}{2} \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \quad , \tag{11}$$

so that the normal form for the wave equation is

$$\frac{\partial^2 A^i}{\partial \xi \partial \eta} \; = \; 0 \quad \Rightarrow \quad A^i \; = \; f_1(\xi) + f_2(\eta) \quad . \tag{12}$$

This is a solution of counter-propagating waves, both at speed c.

Now let us ascertain the vector properties of the waves associated with the electromagnetic fields. For propagation along the x-direction, if the Coulomb gauge condition is invoked, $\nabla \cdot \mathbf{A} = 0$, then since $A_i = A_i(x)$, it collapses to

$$\frac{\partial A_x}{\partial x} \; = \; 0 \quad \Rightarrow \quad A_x(x) \; = \; A_x(t) \quad . \tag{13}$$

If we also set $\phi = 0$, WLOG, insertion into the wave equation then gives

$$\frac{\partial^2 A_x}{\partial^2 t} \; = \; 0 \quad \Rightarrow \quad A_x(x) \; = \; \alpha + \beta t \quad \Rightarrow \quad \mathbf{E} \; = \; -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \; = \; -\frac{\beta}{c} \hat{x} \quad . \tag{14}$$

A constant longitudinal electric field is irrelevant to the wave, so it can be set to zero. We can then set $A_x = 0$ (i.e., $\alpha = 0 = \beta$). Therefore we can *choose* the vector potential to have just components A_y and A_z perpendicular to the direction of propagation. We will consider just the case of propagation along the positive x-axis, and set

$$A_y \;=\; A_y(t - x/c) \quad , \quad A_z \;=\; A_z(t - x/c) \tag{15}$$

All field functions will depend only on the offset time $\tau = t - x/c$. A Fourier decomposition of the waveform can be posited:

$$\mathbf{A}(x,\, t) \;=\; \int \mathbf{A}_{\mathbf{k}}(\mathbf{k},\, \omega)\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\, \frac{d^4k}{(2\pi)^4} \quad . \tag{16}$$

Only the components $A_y,\, A_z$ in the transverse directions, the $(y,\, z)$-plane, are non-zero, and clearly $\mathbf{k} = k\mathbf{n}$, where the unit vector in the x-direction is denoted by $\mathbf{n}$. It is immediately obvious that

$$k \;\equiv\; |\mathbf{k}| \;=\; \frac{\omega}{c} \tag{17}$$

in order for this Fourier decomposition to satisfy the wave equation solution. Thus, $\mathbf{A}_{\mathbf{k}}(\mathbf{k},\, \omega) \propto \delta(k - \omega/c)$.

The time-dependent electric field vector is simply specified, and because $\mathbf{A}$ possesses only components transverse to $\mathbf{n}$, so also does $\mathbf{E}$:

$$\mathbf{E}(x,\, t) \;=\; -\frac{1}{c}\,\frac{\partial \mathbf{A}}{\partial t} \;=\; i \int \frac{\omega}{c}\, \mathbf{A}_{\mathbf{k}}(\mathbf{k},\, \omega)\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\, \frac{d^4k}{(2\pi)^4} \quad . \tag{18}$$

Prescribing the wave magnetic field is only slightly more involved. The Fourier expression yields the identity ($\partial \mathbf{A}/\partial y = \partial \mathbf{A}/\partial z$)

$$\mathbf{n} \times \frac{\partial \mathbf{A}}{\partial t} \;=\; -c\,\mathbf{n} \times \frac{\partial \mathbf{A}}{\partial x} \;\equiv\; -c\,\nabla \times \mathbf{A} \;=\; -c\,\mathbf{B} \quad , \tag{19}$$

remembering that only the transverse components of $\mathbf{A}$ are non-zero. Remembering the definition of $\mathbf{E}$ for our wave [see Eq. (18)], this reduces to

$$\mathbf{B} \;=\; \mathbf{n} \times \mathbf{E} \;=\; i \int \frac{\omega}{c} \Big\{ \mathbf{n} \times \mathbf{A}_{\mathbf{k}}(\mathbf{k},\, \omega) \Big\}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\, \frac{d^4k}{(2\pi)^4} \quad . \tag{20}$$

Therefore, $\mathbf{B}$ is perpendicular to both $\mathbf{n}$ and $\mathbf{E}$. Moreover, it follows from this vector triad construction that $|\mathbf{B}| = |\mathbf{E}|$.

• We have therefore finally established what we have alluded to for quite a while now: electromagnetic waves are transverse phenomena, with both field components orthogonal to the direction of propagation, and to each other, at least at all points in time and space. In addition, *the two fields are of equal magnitude*, as necessitated by light being massless in the theory of relativity.

• These idealized properties, together with the clean **dispersion relation** $\omega = kc$, are relinquished in material media, where charges and currents are finite, and the wave equation is modified, and disturbances propagate at speeds less than c; moreover, *they are not exactly transverse*!

The Poynting vector or energy flux of the plane wave can now be specified:

$$\mathbf{S} \; = \; \frac{c}{4\pi}\,\mathbf{E}\times\mathbf{B} \; = \; \frac{c}{4\pi}\,\mathbf{E}\times\big(\mathbf{n}\times\mathbf{E}\big) \; = \; \frac{c}{4\pi}\Big\{\big(\mathbf{E}\cdot\mathbf{E}\big)\,\mathbf{n} - \cancel{\big(\mathbf{n}\cdot\mathbf{E}\big)}\,\mathbf{E}\Big\}\;, \tag{21}$$

so that it follows that

$$\mathbf{S} \; = \; \frac{c}{4\pi}\,E^2\,\mathbf{n} \; = \; \frac{c}{4\pi}\,B^2\,\mathbf{n} \quad . \tag{22}$$

Accordingly, the energy of an electromagnetic wave flows along $\mathbf{k}$, which is called its **wavevector**. The energy density $U_{\rm em}$ of the wave is related to the Poynting vector:

$$U_{\rm em} \; = \; \frac{E^2+B^2}{8\pi} \; = \; \frac{E^2}{4\pi} \quad \Rightarrow \quad \mathbf{S} \; = \; cU_{\rm em}\,\mathbf{n} \quad , \tag{23}$$

so that $|\mathbf{S}| = U_{\rm em}c$, and the momentum per unit volume is $\mathbf{S}/c^2$. The **flux of momentum** is determined also from the energy-momentum tensor, and since our wave travels along the x-direction, it is just $T^{xx} = -\sigma_{xx} = U_{\rm em}$.

• For a Lorentz boost between observers viewing the wave, say in the K and K' frames, we have determined [in homework] that

$$U_{\rm em} \; = \; \gamma^2\Big(U'_{\rm em} + 2\beta\,\frac{S'_x}{c} + \beta^2\sigma'_{xx}\Big) \quad . \tag{24}$$

For a wave propagation angle θ' relative to the direction of the boost, the result from the energy-momentum tensor is

$$U_{\rm em} \; = \; \gamma^2 U'_{\rm em}\big(1+\beta\cos\theta'\big)^2 \quad . \tag{25}$$

Here one **Doppler factor** $\gamma(1+\beta\cos\theta')$ is associated with the blueshift of the light, and the other with length contraction of the volume, or, equivalently, one for each power of the electric/magnetic fields.

1.2 Monochromatic Plane Waves

The special case where the wavenumber and frequency of the vector potential Fourier transform are infinitely narrow, i.e. a delta function, defines a **monochromatic wave**. Thus **L&L Sec. 48**

$$\mathbf{A}(x,\, t) \;=\; \int \mathbf{A}_{\mathbf{k}}(\mathbf{k},\, \omega)\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\, \frac{d^4k}{(2\pi)^4} \;\rightarrow\; \mathbf{A_0}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \quad . \tag{26}$$

As before, the non-dispersive restriction gives a phase speed ω/k of the wave of c, and the wavenumber and the **wavelength** λ are related by

$$\lambda \;=\; \frac{2\pi}{k} \quad , \quad k \;=\; |\mathbf{k}| \;=\; \frac{\omega}{c} \quad . \tag{27}$$

In this case, the electric field (time derivative) and the magnetic field (spatial gradient) are simply expressed:

$$\mathbf{E} \;=\; ik\,\mathbf{A} \quad , \quad \mathbf{B} \;=\; i\mathbf{k}\times\mathbf{A} \quad . \tag{28}$$

In general, we can work algebraically with complex quantities, however, when it is time to determine real fields, we set

$$\mathbf{E} \;=\; \mathrm{Re}\Big(\mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\Big) \quad , \quad \mathbf{B} \;=\; \mathrm{Re}\Big(\mathbf{B}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\Big) \tag{29}$$

Here $\mathbf{E}_0 \;=\; ik\mathbf{A_0}$ and $\mathbf{B}_0 \;=\; i\mathbf{k}\times\mathbf{A_0}$. The signature character of the monochromatic wave is that both fields vary purely sinusoidally and with correlated phases. To determine the real part, we write $\mathbf{E}_0 = (\mathbf{E}_1 + i\mathbf{E}_2)\, e^{-i\alpha}$, where $\mathbf{E}_1$ and $\mathbf{E}_2$ are real vectors. If we demand that $(\mathbf{E}_1 + i\mathbf{E}_2)^2$ be a real quantity, then $\mathbf{E}_1 \cdot \mathbf{E}_2 = 0$, i.e. they are perpendicular. Then, we can set $\mathbf{E}_1 = E_1\hat{y}$ and $\mathbf{E}_2 = E_2\hat{z}$. The real components of the electric field are

$$E_y \;=\; E_1\, \cos\!\big(\omega t - \mathbf{k}\cdot\mathbf{r} + \alpha\big) \quad , \quad E_z \;=\; \pm E_2\, \sin\!\big(\omega t - \mathbf{k}\cdot\mathbf{r} + \alpha\big) \quad . \tag{30}$$

The choice of sign for the E_z component determines the sense of this component relative to the E_y component, and thus defines the positive or negative **helicity** of the wave. If the wave is generated by a charge gyrating in a magnetic field, then the helicity is determined by the sign of the charge.

Eliminating the explicit time dependence gives

$$\frac{E_y^2}{E_1^2} + \frac{E_z^2}{E_2^2} = 1 \quad . \tag{31}$$

Therefore, as time progresses, the electric vector rotates in the plane orthogonal to the direction of propagation, with its tip tracing out an ellipse in the (y, z)-plane. This is the most general form of a monochromatic wave, and it is said to be **elliptically polarized**.

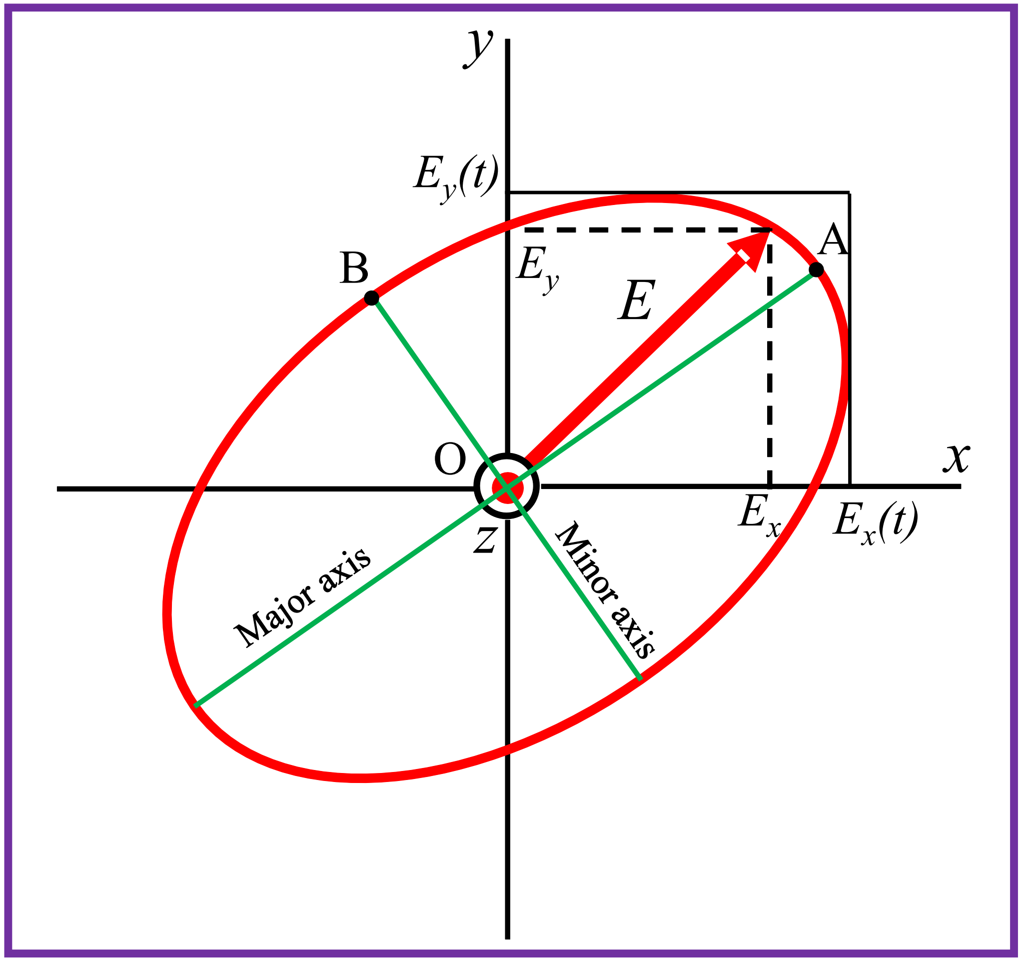
$\boxed{\textbf{Plot:}}$ The Polarization Ellipse

- If either of E_1 or E_2 is zero, then the wave is described by just a single sinusoid, and is said to be **linearly polarized**. This is the "purest" polarization configuration for electromagnetic waves, and can be generated by an alternating current in a wire: the electric field vector is confined to a plane that contains the **k** vector also.

- In the special case that $E_1 = E_2 \neq 0$, then the ellipse reduces to a circle and the wave is said to be **circularly polarized**. These are a superposition of two linearly polarized waves of equal **electric amplitude** but orthogonal polarization (i.e., **E**-field vector direction), and a single phase offset.

∗ *Elliptically polarized* waves are a superposition of two linearly polarized waves of unequal amplitude, orthogonal polarization, and a phase offset.

The Polarization Ellipse



- The elliptical path in the (x,y) plane of the electric field vector ***E*** for an electromagnetic wave propagating in the z-direction.
- From Fig. 2.14 of A. Pal, PhD Thesis, Swansea Univ. (2013).