# 4 Magnetic Field Configurations

Now we extend the static field considerations to include magnetic fields. These require currents. The considerations will be restricted to time-averaged cases, where charge motions are relatively ordered, and the contributions from time-varying electric fields are small. This means that we are not considering electromagnetic waves, yet we will very soon.

## 4.1 Biot-Savart Law

The two Maxwell's equations that control the magnetic field are **L&L Sec. 43**

$$\nabla \cdot \mathbf{B} \;=\; 0 \quad , \quad \nabla \times \mathbf{B} \;=\; \frac{1}{c}\,\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\,\mathbf{j} \quad . \tag{52}$$

Over long time periods, these may be averaged using our standard protocols, and the varying electric field term becomes insignificant:

$$\left\langle \frac{\partial \mathbf{E}}{\partial t} \right\rangle_t \;\equiv\; \frac{1}{T}\int_0^T \frac{\partial \mathbf{E}}{\partial t}\, dt \;\to\; 0 \quad ; \tag{53}$$

electric fields are minuscule on large timescales. Again we are neglecting the possibility of electromagnetic waves. The time-averaged Maxwell equations become

$$\nabla \cdot \langle \mathbf{B} \rangle_t \;=\; 0 \quad , \quad \nabla \times \langle \mathbf{B} \rangle_t \;=\; \frac{4\pi}{c}\,\langle \mathbf{j} \rangle_t \quad . \tag{54}$$

These will be manipulated. Hereafter, the time-averaging notation will be dropped, though it will be implicitly assumed in the ensuing forms.

Introducing the time-averaged vector potential, then $\mathbf{B} \to \nabla \times \mathbf{A}$, and Gauss' law is trivial. Ampere's circuital law assumes the interesting form:

$$\nabla \times \big(\nabla \times \mathbf{A}\big) \;=\; \nabla\big(\nabla \cdot \mathbf{A}\big) - \nabla^2 \mathbf{A} \;=\; \frac{4\pi}{c}\,\mathbf{j} \quad . \tag{55}$$

The first term on the left can be eliminated by adopting the **Coulomb gauge** with $\nabla \cdot \mathbf{A} = 0$. Then

$$\nabla^2 \mathbf{A} \;=\; -\frac{4\pi}{c}\,\mathbf{j} \quad . \tag{56}$$

This is satisfyingly simple in that it is a vector analog of Poisson's equation, with the scalar potential $\phi$ being replaced by the vector potential $\mathbf{A}$, and the charge density $\rho$ being replaced by the current density $\mathbf{j}/c$. One can then integrate the Laplacian as was done for Coulomb's law, and derive

$$\mathbf{A} \;=\; \frac{1}{c}\int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r' \;\equiv\; \frac{1}{c}\sum_n \frac{q_n \mathbf{v}_n}{|\mathbf{r}-\mathbf{x}_n|} \tag{57}$$

for the time-averaged vector potential in terms of the time-averaged current. Here $\mathbf{r}$ is the position vector from any point within the integration volume to where the field is measured. The discrete charge alternative form looks just like the ensemble form for Coulomb's law, but now with charge $q_n$ replaced by current $q_n\mathbf{v}_n$.

The time-averaged magnetic field can now be derived and simplified:

$$\begin{aligned} \mathbf{B} \;&=\; \nabla_{\mathbf{r}} \times \left( \frac{1}{c}\int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r' \right) \\ &=\; \frac{1}{c}\int \left\{ \left( \nabla_{\mathbf{r}} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) \times \mathbf{j}(\mathbf{r}') + \frac{1}{|\mathbf{r}-\mathbf{r}'|} \cancelto{\mathbf{0}}{\left( \nabla_{\mathbf{r}} \times \mathbf{j}(\mathbf{r}') \right)} \right\} d^3r' \quad . \end{aligned} \tag{58}$$

Here we have used a standard vector identity for the curl of the product of a scalar and a vector. The curl operates only on the position vector $\mathbf{r}$ (hence the subscript) out to where the field is measured, and so $\nabla_{\mathbf{r}} \times \mathbf{j}(\mathbf{r}') = \mathbf{0}$. Therefore, the time-averaged magnetic field satisfies

$$\mathbf{B} \;=\; \frac{1}{c}\int \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, d^3r' \quad , \tag{59}$$

since $\nabla_{\mathbf{r}}(1/|\mathbf{r}-\mathbf{r}'|) \;=\; -(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|^3$. Carefully note the order of the cross product. Hereafter the explicit use of subscript $\mathbf{r}$ on the $\nabla$ operator is dropped. This result, which bears considerable semblance to the magnetic field of a moving charge, is known as the **Biot-Savart Law**, and only applies in a time-averaged sense. Such steady configurations are commonly realized in current systems, *for example coils and current loops.*

- For magnetostatic configurations, $\partial\rho/\partial t = 0 \Rightarrow \nabla\cdot\mathbf{j} = 0$, so that $\mathbf{j}$ can be expressed as a curl. Eq. (54) gives $\mathbf{j} = (c/4\pi)\,\nabla\times\mathbf{B}$ as the inversion of the Biot-Savart law. This should be obvious when forming $\nabla\times\mathbf{B}$ as the integrand is then proportional to $\nabla\times\nabla(1/|\mathbf{r}-\mathbf{r}'|) = 4\pi\delta^3(\mathbf{r}-\mathbf{r}')$, and the evaluation of the integral is trivial.

## 4.2 Magnetic Moments

The Biot-Savart Law is expressed via a <u>continuum</u> form for the current vector. **L&L**
In this Section we explore the discrete form in a search for the magnetic **Sec. 44**
analog of the electric dipole moment formula. Thus a far-field expansion is *de rigueur*, and we perform a Taylor series expansion.

$$\mathbf{A} \;=\; \frac{1}{c}\sum_n \frac{q_n \mathbf{v}_n}{|\mathbf{r}-\mathbf{x}_n|} \;\approx\; \frac{1}{c\,|\mathbf{r}|}\sum_n q_n \mathbf{v}_n - \frac{1}{c}\sum_n q_n \mathbf{v}_n \left\{ \mathbf{x}_n \cdot \nabla\left(\frac{1}{r}\right)\right\} \quad . \tag{60}$$

The first term is proportional to the time derivative of the electric dipole moment:

$$\sum_n q_n \mathbf{v}_n \;=\; \frac{d}{dt}\left(\sum_n q_n \mathbf{x}_n\right) \;=\; \langle \dot{\mathbf{d}} \rangle_t \;\to\; \mathbf{0} \quad . \tag{61}$$

In general, $\dot{\mathbf{d}} \neq \mathbf{0}$. However, here we are considering long-term time averages, and so the integrated derivative tends to zero, as before. Thus, only the second term contributes:

$$\mathbf{A} \;\approx\; \frac{1}{cr^3}\sum_n q_n \mathbf{v}_n \left(\mathbf{x}_n \cdot \mathbf{r}\right) \quad . \tag{62}$$

Since $\mathbf{v}_n = d\mathbf{x}_n/dt$, this can be re-expressed using the chain rule of differentiation to symmetrize the roles of $\mathbf{x}_n$ and its derivative

$$\begin{aligned} \cancel{\frac{1}{2}\frac{d}{dt}\left\{\sum_n q_n \left(\mathbf{x}_n \cdot \mathbf{r}\right)\mathbf{x}_n\right\}} \;&=\; \sum_n q_n \mathbf{v}_n \left(\mathbf{x}_n \cdot \mathbf{r}\right) \\ &\quad -\frac{1}{2}\sum_n q_n \Big\{ \left(\mathbf{x}_n \cdot \mathbf{r}\right)\mathbf{v}_n - \left(\mathbf{v}_n \cdot \mathbf{r}\right)\mathbf{x}_n \Big\} \;. \end{aligned} \tag{63}$$

The long-term time average of the time derivative on the LHS is also zero, and so we can replace the first line by the negative of the second in the expression for $\mathbf{A}$. It follows that the time-averaged vector potential is

$$\begin{aligned} \mathbf{A} \;&\approx\; \frac{1}{2cr^3}\sum_n q_n \Big\{ \left(\mathbf{x}_n \cdot \mathbf{r}\right)\mathbf{v}_n - \left(\mathbf{v}_n \cdot \mathbf{r}\right)\mathbf{x}_n \Big\} \\ &=\; \frac{1}{2cr^3}\sum_n q_n \Big\{ \left(\mathbf{x}_n \times \mathbf{v}_n\right) \times \mathbf{r} \Big\} \quad . \end{aligned} \tag{64}$$

Recognizing that the anti-symmetric difference is just a triple cross product simplifies the expression, and guides the next step in the development.

This form naturally leads to the definition for the **magnetic moment**:

$$\boldsymbol{\mu} \;=\; \frac{1}{2c}\sum_n q_n \big\langle \mathbf{x}_n \times \mathbf{v}_n \big\rangle_t \quad , \tag{65}$$

where we have explicitly included the time-average notation. Then the vector potential assumes a compact form:

$$\mathbf{A} \;=\; \frac{\boldsymbol{\mu} \times \mathbf{r}}{r^3} \quad . \tag{66}$$

This is very useful for deriving $\mathbf{A}$ values for various current configurations.

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The magnetic field is now simply obtained using standard vector identities:

$$\begin{aligned} \mathbf{B} \;&\approx\; \nabla \times \left( \frac{\boldsymbol{\mu} \times \mathbf{r}}{r^3} \right) \\ &=\; \cancel{\left( \frac{\mathbf{r}}{r^3} \cdot \nabla \right)} \boldsymbol{\mu} - \left( \boldsymbol{\mu} \cdot \nabla \right) \frac{\mathbf{r}}{r^3} + \boldsymbol{\mu}\, \nabla \cdot \left( \frac{\mathbf{r}}{r^3} \right) - \frac{\mathbf{r}}{r^3} \cancel{\nabla \cdot \boldsymbol{\mu}} \\ &=\; \boldsymbol{\mu}\, \cancel{\nabla \cdot \left( \frac{\mathbf{r}}{r^3} \right)} - \left( \boldsymbol{\mu} \cdot \nabla \right) \frac{\mathbf{r}}{r^3} \quad . \end{aligned} \tag{67}$$

The space derivatives of $\boldsymbol{\mu}$ are both zero since they pertain to the point where the field is measured, and not to $\boldsymbol{\mu}$ itself. In addition,

$$\nabla \cdot \left( \frac{\mathbf{r}}{r^3} \right) \;=\; \mathbf{r} \cdot \nabla \left( \frac{1}{r^3} \right) + \frac{1}{r^3}\, \nabla \cdot \mathbf{r} \;=\; 0 \quad , \tag{68}$$

a result that can be inferred from the Coulomb electric field, which has zero divergence at far-field points where charges are not present. Next,

$$\left( \boldsymbol{\mu} \cdot \nabla \right) \frac{\mathbf{r}}{r^3} \;=\; \frac{1}{r^3} \left( \boldsymbol{\mu} \cdot \nabla \right) \mathbf{r} + \mathbf{r} \left( \boldsymbol{\mu} \cdot \nabla \right) \frac{1}{r^3} \quad . \tag{69}$$

Therefore, using $\mathbf{n} = \mathbf{r}/r$ , one arrives at the final result for the time-averaged magnetic field:

$$\mathbf{B} \;=\; \frac{3(\mathbf{n} \cdot \boldsymbol{\mu})\, \mathbf{n} - \boldsymbol{\mu}}{r^3} \quad . \tag{70}$$

This is the same formula as for the electric dipole, but with $\mathbf{d} \to \boldsymbol{\mu}$ . This field configuration is therefore referred to as defining a **magnetic dipole**.

• Using the angle notation $\cos\theta = \hat{\boldsymbol{\mu}} \cdot \hat{\mathbf{n}}$, one can express the field components parallel to and perpendicular to the magnetic moment $\boldsymbol{\mu}$:

$$B_{\parallel} \;=\; \frac{\mu}{r^3}\Big(3\cos^2\theta - 1\Big) \quad , \quad B_{\perp} \;=\; \frac{\mu}{r^3}\Big(3\sin\theta\cos\theta\Big) \quad . \tag{71}$$

Again, these can be easily rotated to develop the polar coordinate form for the radial and tangential components of the dipole field:

$$\begin{aligned} B_r \;&\equiv\; B_{\parallel}\cos\theta + B_{\perp}\sin\theta \;=\; \frac{2\mu\,\cos\theta}{r^3} \\ B_\theta \;&\equiv\; B_{\parallel}\sin\theta - B_{\perp}\cos\theta \;=\; -\frac{\mu\sin\theta}{r^3} \quad . \end{aligned} \tag{72}$$

• Since the magnetic dipole form is derived in the far-field approximation, it cannot be applied to find the field configuration near or inside a solenoid.

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An insightful special case is that where all the charges in the system have *the same mass-to-charge ratio*, i.e. $q_n/m_n = q/m$ for some average charge $q$ and average mass $m$. Then the magnetic moment can be expressed as

$$\boldsymbol{\mu} \;\equiv\; \frac{1}{2c}\sum_n q_n \mathbf{x}_n \times \mathbf{v}_n \;=\; \frac{q}{2mc}\sum_n \mathbf{x}_n \times \big(m_n \mathbf{v}_n\big) \quad . \tag{73}$$

For non-relativistic charge motions, $m_n\mathbf{v}_n = \mathbf{p}_n$ is the linear momentum of a charge, and so

$$\boldsymbol{\mu} \;=\; \frac{q}{2mc}\sum_n \mathbf{x}_n \times \mathbf{p}_n \;=\; \frac{q}{2mc}\,\boldsymbol{\mathcal{L}} \quad . \tag{74}$$

where $\boldsymbol{\mathcal{L}}$ is the total angular momentum of the system. In this special non-relativistic case, the magnetic dipole moment is proportional to the angular momentum, and the constant of proportionality $q/(2mc)$ is known as the **classical magneton**.

• This coupling between rotating charge configurations and magnetic dipoles sets the scene for the interpretation of spin in quantum mechanical systems. In particular, Bohr's quantization condition $|\boldsymbol{\mathcal{L}}| = n\hbar$ for an atom's electronic states yields the **quantum magneton** $e\hbar/(2m_e c)$ for the electron.

## 4.3 Larmor Precession

Now immerse our system of charges/currents in an external magnetic field, which will be assumed to be uniform on the scale of the system. The time-averaged Lorentz force equation is **L&L Sec. 45**

$$\langle \mathbf{F} \rangle_t \; = \; \frac{1}{c} \sum_n q_n \langle \mathbf{v}_n \times \mathbf{B} \rangle_t \; = \; \frac{d}{dt} \left\{ \sum_n q_n \langle \mathbf{x}_n \times \mathbf{B} \rangle_t \right\} \; \to \; \mathbf{0} \quad . \tag{75}$$

Again, since long-term time averages of derivatives are zero, then so is the force, since it is a pure time derivative. This is because the long-term average integrates over the entire volume, and there is no energy gain or loss since the magnetic field does no net work.

Yet quantities derived from the ensemble force are not necessarily zero. An example is provided by the magnetic **torque**:

$$\boldsymbol{\tau} \; = \; \frac{1}{c} \sum_n q_n \Big\langle \mathbf{x}_n \times \big( \mathbf{v}_n \times \mathbf{B} \big) \Big\rangle_t \quad . \tag{76}$$

The triple vector product can be expanded (averaging notation dropped):

$$\begin{aligned} \boldsymbol{\tau} \; &= \; \frac{1}{c} \sum_n q_n \Big\{ \mathbf{v}_n (\mathbf{x}_n \cdot \mathbf{B}) - \mathbf{B} (\mathbf{v}_n \cdot \mathbf{x}_n) \Big\} \\ &= \; \frac{1}{c} \sum_n q_n \Big\{ \mathbf{v}_n (\mathbf{x}_n \cdot \mathbf{B}) - \frac{1}{2} \mathbf{B} \cancel{\frac{d}{dt} (\mathbf{x}_n^2)} \Big\} \quad . \end{aligned} \tag{77}$$

The time-averaging of the second term is likewise zero. One can then employ the chain rule derivative manipulation used in deriving the magnetic moment to anti-symmetrize the sum:

$$\boldsymbol{\tau} \; = \; \frac{1}{c} \sum_n q_n \mathbf{v}_n \big( \mathbf{x}_n \cdot \mathbf{B} \big) \; = \; \frac{1}{2c} \sum_n q_n \Big\{ \mathbf{v}_n \big( \mathbf{x}_n \cdot \mathbf{B} \big) - \mathbf{x}_n \big( \mathbf{v}_n \cdot \mathbf{B} \big) \Big\} \quad . \tag{78}$$

It then follows that the torque possesses a simple form:

$$\boldsymbol{\tau} \; = \; \left\{ \frac{1}{2c} \sum_n q_n \Big( \mathbf{x}_n \times \mathbf{v}_n \Big) \right\} \times \mathbf{B} \; = \; \boldsymbol{\mu} \times \mathbf{B} \quad . \tag{79}$$

This is a magnetic analog of the equivalent electric dipole result, $\boldsymbol{\tau} = \mathbf{d} \times \mathbf{E}$. The torque-free case is when $\boldsymbol{\mu}$ is aligned or counter-aligned with the field. Otherwise, the finite torque, being orthogonal to $\boldsymbol{\mu}$, will rotate the magnetic dipole, just as a spinning top precesses due to a gravitational torque.

To see this more clearly, the torque is just the rate of change of the angular momentum $\boldsymbol{\mathcal{L}}$ of the system: $\boldsymbol{\tau} = d\boldsymbol{\mathcal{L}}/dt$. Now consider a system of charges with equal charge to mass ratios. Then $q_n/m_n = q/m$ for some average charge $q$ and average mass $m$. As before, the magnetic moment is proportional to the angular momentum, so that

$$\frac{d\boldsymbol{\mathcal{L}}}{dt} \;=\; \boldsymbol{\tau} \;=\; \frac{q}{2mc}\,\boldsymbol{\mathcal{L}}\times\mathbf{B} \;=\; \boldsymbol{\mathcal{L}}\times\boldsymbol{\Omega} \quad \text{for} \quad \boldsymbol{\Omega} \;=\; \frac{q}{2mc}\,\mathbf{B} \quad . \tag{80}$$

This vector ODE has sinusoidal solutions for the components transverse to $\boldsymbol{\Omega}$, leaving the magnitude of $\boldsymbol{\mathcal{L}}$ constant in time. It is an analog of

$$\frac{d\mathbf{v}}{dt} \;=\; \frac{q}{mc}\,\mathbf{v}\times\mathbf{B} \quad , \tag{81}$$

the Lorentz force for a non-relativistic charge. Therefore, the magnetic moment vector $\boldsymbol{\mu}$ **precesses** about $\mathbf{B}$ at a Larmor frequency $eB/(2mc)$, with the tip of the vector tracing out a circle orthogonal to $\mathbf{B}$. *Observe that the Larmor frequency is exactly half the cyclotron frequency*!

- This **Larmor precession** does not change the strength or energy of the magnetic dipole. The energy of the magnetic dipole in a uniform external field can be derived from the Lagrangian $L = e\mathbf{A}\cdot\mathbf{v}/c$. The vector potential in this case is just $\mathbf{A} = (\mathbf{B}\times\mathbf{r})/2$. Adapting this for the charge ensemble,

$$\begin{aligned} L_{\mathrm{B}} \;&=\; \frac{1}{c}\sum_n q_n\mathbf{A}_n\cdot\mathbf{v}_n \;=\; \frac{1}{2c}\sum_n q_n\big(\mathbf{B}\times\mathbf{x}_n\big)\cdot\mathbf{v}_n \\ &=\; \frac{1}{2c}\sum_n q_n\big(\mathbf{x}_n\times\mathbf{v}_n\big)\cdot\mathbf{B} \quad , \end{aligned} \tag{82}$$

using a familiar vector cross product identity. Consequently

$$U_{\mathrm{B}} \;=\; -L_{\mathrm{B}} \;=\; -\boldsymbol{\mu}\cdot\mathbf{B} \quad . \tag{83}$$

This is a magnetic analog of the electric dipole result $U_{\mathrm{E}} = -\mathbf{d}\cdot\mathbf{E}$. The precession of the magnetic moment clearly leaves $\boldsymbol{\mu}\cdot\mathbf{B}$ invariant, and so it conserves energy: no work is done by the external field.

This result *implicitly assumes no long term time changes in far-field locales*, which omits the contribution of electromagnetic radiation by accelerating charges that generate $\boldsymbol{\mu}$. Thus, in reality, even in the absence of an external magnetic field $\mathbf{B}$, a spinning magnetic dipole will loss energy.