which can be substituted directly into the Coulomb form. The vector potential in the K frame can be similarly prescribed using $\mathbf{A} = \gamma\phi'\mathbf{v}/c = \phi\mathbf{v}/c$, since $\mathbf{A}' = 0$. If one then sets

$$\left(r^*\right)^2 \;\equiv\; \frac{(r')^2}{\gamma^2} \;=\; (x-\beta ct)^2 + \left(1-\beta^2\right)(y^2+z^2) \quad , \tag{20}$$

then the components of the four-potential in the K frame are

$$\phi \;=\; \frac{q}{r^*} \quad , \quad \mathbf{A} \;=\; \frac{q\,\mathbf{v}}{cr^*} \;=\; \frac{q\beta}{r^*}\,\hat{x} \quad . \tag{21}$$

The electric field can be obtained in either of two ways. One path is to use the Coulomb field form in the charge's rest frame and Lorentz transform it using our standard boost relations. This gives

$$E_x \;=\; E'_x \;=\; \frac{qx'}{(r')^3} \quad , \quad E_y \;=\; \gamma E'_y \;=\; \frac{\gamma q y'}{(r')^3} \quad , \quad E_z \;=\; \frac{\gamma q z'}{(r')^3}\,. \tag{22}$$

This is the simplest approach, yet the same equations will result by taking the 4-gradient of the 4-potential $(\phi, \mathbf{A})$ using the expression for r' in Eq. (19). The field can then be written in vector notation:

$$\mathbf{E} \;\equiv\; -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \;=\; \frac{q\,\mathbf{r}}{\gamma^2(r^*)^3} \quad \text{for} \quad \mathbf{r} \;=\; (x-\beta ct,\, y, z) \quad . \tag{23}$$

This is the first indication that there is a **retardation** or time delay in establishing the field: causality applies in communicating electromagnetic information from a moving charge.

If the radius vector $\mathbf{r}$ makes an angle θ to the boost direction, then clearly $y^2 + z^2 = r^2\sin^2\theta$ and $(x-\beta ct)^2 = r^2\cos^2\theta$. This leads to a re-expression for the electric field vector form:

$$\mathbf{E} \;=\; \frac{q\,\mathbf{r}}{\gamma^2 r^3}\left(1-\beta^2\sin^2\theta\right)^{-3/2} \quad \text{since} \quad (r^*)^2 = r^2\left(1-\beta^2\sin^2\theta\right) \quad . \tag{24}$$

This highlights the angular dependence of the electric field relative to the boost direction. The two extreme cases are parallel ($\theta = 0$) to and perpendicular to ($\theta = \pi/2$) the boost:

$$E_\parallel \;\equiv\; \left|\mathbf{E}(\theta=0)\right| \;=\; \frac{1}{\gamma^2}\,\frac{q}{r^2} \quad , \quad E_\perp \;\equiv\; \left|\mathbf{E}(\theta=\pi/2)\right| \;=\; \gamma\,\frac{q}{r^2} \tag{25}$$

The perpendicular field is enhanced by the boost's γ factor, a general property of Lorentz transformations. The parallel case is depressed by $1/\gamma^2$. The dividing boundary between these two asymptotic cases is $\theta \sim \pi/2 - 1/\gamma$ so that *the electric field is concentrated in a* **Lorentz disc**.

Plot: Field Compression Perpendicular to the Boost Direction

• The angular size of the disk just reflects a *conservation of electric flux*: the quasi-parallel domain is expanded in solid angle by the boost and so the field therein is depressed, while the enhanced quasi-perpendicular domain is reduced in solid angle by $\sim 1/\gamma^2$ so that the field magnitude must be increased by $\sim \gamma$. This character exactly captures length contraction along the boost $\boldsymbol{\beta}$ direction from K' to K.

Since $\mathbf{B}' = \mathbf{0}$, the magnetic field in the K system can be quickly obtained from the Lorentz transformation formulae as a drift-like field:

$$\mathbf{B} \;=\; +\gamma \frac{\mathbf{v}}{c} \times \mathbf{E}' \;=\; \boldsymbol{\beta} \times \big(E_y \hat{y} + E_z \hat{z} \big) \;=\; \frac{\beta}{\gamma^2} \, \frac{q}{(r^*)^3} \, \hat{x} \times \mathbf{r} \quad , \qquad (26)$$

it has only components perpendicular to the boost.

It is now a simple matter to determine the Lorentz force between two charges q_1 and q_2 moving with the same velocity $\mathbf{v}$ in K. This is

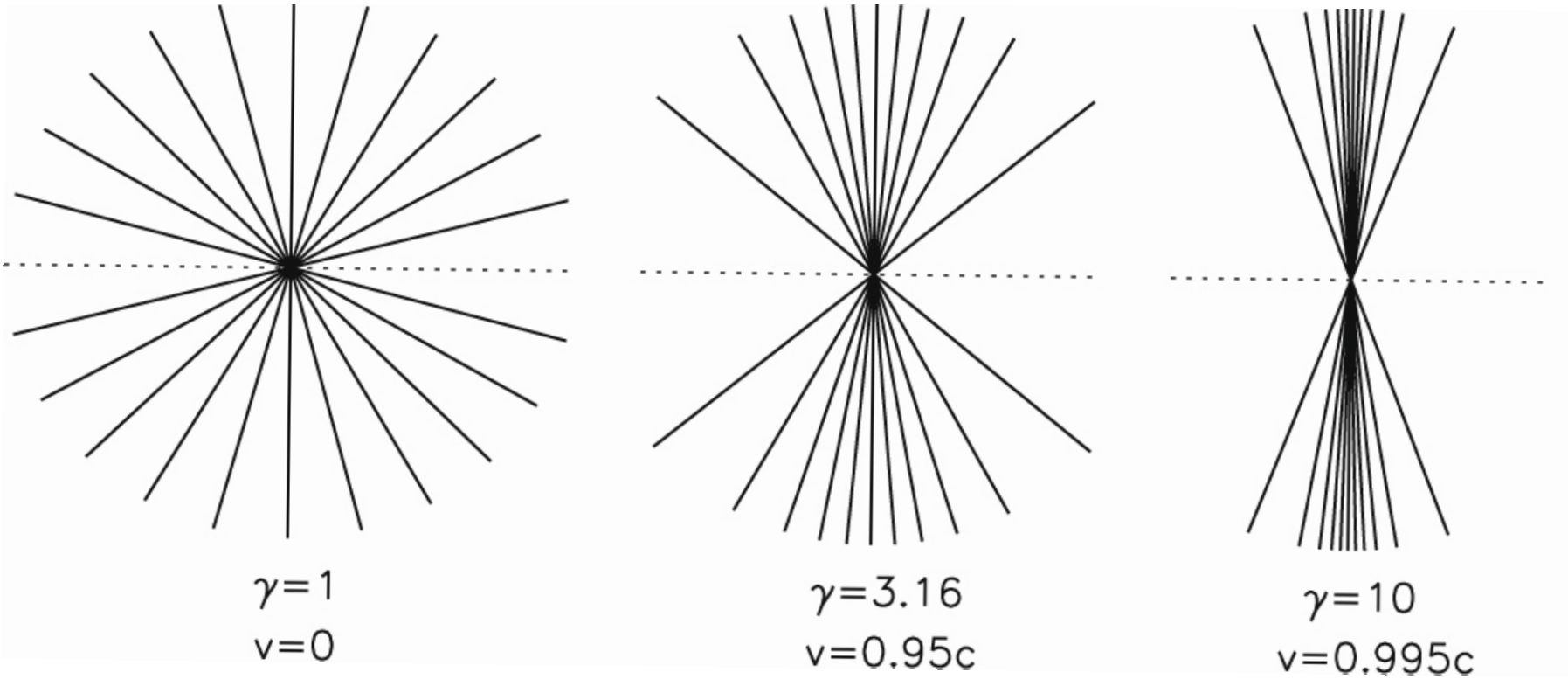
$$\mathbf{F} \;\equiv\; \frac{d\mathbf{p}}{dt} \;=\; q_1 \left\{ \mathbf{E} + \frac{\mathbf{v}}{c} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{E} \right) \right\} \;=\; q_1 \big(1 - \beta^2 \big) \, \mathbf{E} + q_1 \big(\boldsymbol{\beta} \cdot \mathbf{E} \big) \, \boldsymbol{\beta} \quad . \qquad (27)$$

Here the standard vector identity $\boldsymbol{\beta} \times (\boldsymbol{\beta} \times \mathbf{E}) = \big(\boldsymbol{\beta} \cdot \mathbf{E} \big) \, \boldsymbol{\beta} - \beta^2 \, \mathbf{E}$ has been used to expand the double cross product. One can now substitute in the trigonometric form in Eq. (24) and derive the components parallel (F_x) and perpendicular (F_y) to the motion:

$$F_x \;=\; \frac{q_1 q_2}{r^2} \, \frac{\cos\theta}{\gamma^2 \big(1 - \beta^2 \sin^2\theta \big)^{3/2}} \quad , \quad F_y \;=\; \frac{q_1 q_2}{r^2} \, \frac{\sin\theta}{\gamma^4 \big(1 - \beta^2 \sin^2\theta \big)^{3/2}} \quad . \qquad (28)$$

An extension of these formulae to unequal velocities can be used in determining differential cross sections for Coulomb collisions between moving charges.

Electric Field of a Moving Charge



- The electric field lines for a charge moving horizontally at three different constant speeds **v**, as labelled. Relativistic "compression" of the field transverse to the motion is evident.
- Adapted from Fig. 3 of Singal, *J. Phys. Comm.*, **4**, 095023 (2020).

3 Electrostatic Multipoles

Most electrostatic systems possess ensembles of charges that are fairly confined in their spatial extent, and are electrostatically neutral. Providing compact formalism for the fields that they generate is expedient both for describing the response of external charges to their fields, and also for elements of electromagnetic radiation. The Taylor series approximation for far-field configurations naturally establishes a **multipolar** or perturbation field construction, the focus of this Section.

3.1 The Dipole Moment

L&L Sec. 40

The lowest order approximation is for an **electrostatic dipole**, essentially equivalent to two equal charges of opposite sign that are spatially separated. Consider a localized distribution of charges from a large distance at point $\mathbf{r}$. If the position of charge q_n is $\mathbf{x}_n$, then the total potential function is

$$\phi(\mathbf{r}) \;=\; \sum_n \frac{q_n}{|\mathbf{r}-\mathbf{x}_n|} \quad . \tag{29}$$

We explore this in the domain $|\mathbf{r}| \gg |\mathbf{x}_n|$. The denominator of the potential function may be expanded in a Taylor series:

$$\phi(\mathbf{r}) \;\approx\; \frac{1}{|\mathbf{r}|}\sum_n q_n - \left(\sum_n q_n \mathbf{x}_n\right)\cdot\nabla\left(\frac{1}{r}\right) \quad . \tag{30}$$

We can therefore identify *two moments of the charge distribution*, the total charge Q and the **dipole moment d**:

$$Q \;=\; \sum_n q_n \quad , \quad \mathbf{d} \;=\; \sum_n q_n \mathbf{x}_n \quad . \tag{31}$$

If the total charge Q of the ensemble is zero, then the dipole moment is independent of the choice for the origin of the coordinates $\mathbf{r}_n \to \mathbf{r}_n + \mathbf{a}$:

$$\mathbf{d} \;\to\; \sum_n q_n \mathbf{x}_n + \mathbf{a}\cancel{\sum_n q_n} \;=\; \mathbf{d} \quad . \tag{32}$$

For such a charge-neutral case, dipole moments for the subsets of positive and negative charges can be identified.

When $Q = 0$, the system is a **pure dipole**, and the gradient function is easily developed:

$$\phi(\mathbf{r}) \;\approx\; \phi_{\rm D} \;=\; -\mathbf{d}\cdot\nabla\Big(\frac{1}{r}\Big) \;=\; \frac{\mathbf{d}\cdot\mathbf{r}}{r^3} \;=\; \frac{d\cos\theta}{r^2} \;=\; \frac{d}{r^2}\,P_1(\cos\theta) \;\;. \qquad (33)$$

Here θ is the angle between the $\mathbf{r}$ vector and the dipole moment, and $P_1(z)$ is a Legendre polynomial. This is a **far-field** result, requiring that $|\mathbf{r}|$ be much larger than the size of the system. The electric field is then simply

$$\mathbf{E} \;=\; -\nabla\Big(\frac{\mathbf{d}\cdot\mathbf{r}}{r^3}\Big) \;=\; -\frac{1}{r^3}\nabla(\mathbf{d}\cdot\mathbf{r}) - (\mathbf{d}\cdot\mathbf{r})\,\nabla\Big(\frac{1}{r^3}\Big) \quad . \qquad (34)$$

If we define $\mathbf{n} = \hat{\mathbf{r}}$ to be the unit vector in the direction of the point of interest [*sketch the geometry*], since $\nabla(1/r^3) = -3\mathbf{n}/r^4$ we can deduce

$$\mathbf{E} \;=\; \frac{3(\mathbf{n}\cdot\mathbf{d})\,\mathbf{n} - \mathbf{d}}{r^3} \qquad (35)$$

for the well-known form for the electric field of a dipole.

• Clearly, the dipolar electric potential $\phi_{\rm D}$ scales as $1/r^2$ at large distances from the ensemble, while the field strength scales as $1/r^3$ there. These are one order smaller than the Coulomb field due to the almost total (but not perfect) cancellation between fields generated by charges of opposite signs that are displaced from each other.

• Using the angle notation $\cos\theta = \hat{\mathbf{d}}\cdot\hat{\mathbf{n}}$, one can express the field components parallel to and perpendicular to the dipole moment $\mathbf{d}$:

$$E_\parallel \;=\; \frac{d}{r^3}\Big(3\cos^2\theta - 1\Big) \quad , \quad E_\perp \;=\; \frac{d}{r^3}\Big(3\sin\theta\cos\theta\Big) \quad . \qquad (36)$$

These can be easily rotated to develop the polar coordinate form for the radial and tangential components of the dipole field:

$$\begin{aligned} E_r \;&\equiv\; E_\parallel\cos\theta + E_\perp\sin\theta \;=\; \frac{2d\,\cos\theta}{r^3} \\ E_\theta \;&\equiv\; E_\parallel\sin\theta - E_\perp\cos\theta \;=\; -\frac{d\sin\theta}{r^3} \quad . \end{aligned} \qquad (37)$$

3.2 Multipole Moments

Higher order contributions are routinely developed. The dot product of a moment like $\mathbf{d}$ and a gradient of $1/r$ extends to capturing higher order moments. Thus, for $\mathbf{r} = (r_1,\, r_2,\, r_3)$, **L&L Sec. 41**

$$\phi(\mathbf{r}) \;=\; \frac{Q}{r} - \mathbf{d}\cdot\nabla\Big(\frac{1}{r}\Big) + \frac{1}{2}\sum_n q_n x_i^n x_j^n \,\frac{\partial^2}{\partial r_i \partial r_j}\Big(\frac{1}{r}\Big) + \ldots \quad . \tag{38}$$

Here the x_i^n are the components of the position of charge q_n , and the second term represents a **quadrupole potential**; it is of 3-tensor character. If the system is charge neutral *and* of zero dipole moment, then this quadrupole term is the leading order contribution.

• An example of a quadrupole charge structure consists of 4 charges of equal magnitude at the vertices of a square, two positive at diagonally opposite vertices, and the two negative ones occupying the other vertices.

The coefficient of the second order differentials is the tensor

$$\frac{1}{2}\sum_n q_n x_i^n x_j^n \quad . \tag{39}$$

This has nine elements, however, only six of these can be independent because the tensor is symmetric under $i \leftrightarrow j$. Moreover, we note that the **Laplacian** of the Coulomb form satisfies

$$\nabla^2\Big(\frac{1}{r}\Big) \;\equiv\; \delta_{ij}\,\frac{\partial^2}{\partial r_i \partial r_j}\Big(\frac{1}{r}\Big) \;=\; 0 \quad . \tag{40}$$

This is just the Poisson equation at positions where there are no charges. This identity restricts the degrees of freedom by one. To see this more completely, now add a term proportional to Eq. (40) to our tensor. So, we can form

$$\phi_{\rm Q} \;=\; \frac{1}{2}\sum_n q_n\Big(x_i^n x_j^n - \frac{x_k^n x_k^n}{3}\,\delta_{ij}\Big)\,\frac{\partial^2}{\partial r_i \partial r_j}\Big(\frac{1}{r}\Big) \tag{41}$$

for the quadrupole correction. This leads naturally to the definition

$$D_{ij} \;=\; \sum_n q_n\Big(3x_i^n x_j^n - x_k^n x_k^n\,\delta_{ij}\Big) \tag{42}$$

for the **quadrupole tensor**. This is symmetric and traceless, i.e. $D_{ii} = 0$, so that it has only 5 independent quantities.

The quadrupole term can now be manipulated by performing the gradient derivatives:

$$\frac{\partial^2}{\partial r_i \partial r_j}\Big(\frac{1}{r}\Big) \;=\; \frac{3 r_i r_j}{r^5} - \frac{\delta_{ij}}{r^3} \quad . \tag{43}$$

The second term here doesn't contribute anything to $\phi_{\rm Q}$ due to the traceless condition $\delta_{ij} D_{ij} = D_{ii} = 0$. Then, identifying $n_i = r_i/r$ for the scaled projections along the coordinate axes, the quadrupole term assumes the form

$$\phi_{\rm Q} \;=\; \frac{D_{ij}}{2}\,\frac{n_i n_j}{r^3} \quad . \tag{44}$$

Since D_{ij} is symmetric and real, i.e. Hermitian, it is diagonalizable, using a change of coordinates (rotation). Furthermore, if it is symmetric about some axis, e.g. the z-axis, then the diagonal form has non-zero components

$$D_{xx} \;=\; D_{yy} \;=\; -\frac{1}{2}\,D_{zz} \quad . \tag{45}$$

The charge configuration is here invariant under rotations of coordinates about the z-axis. In this format, the projections are

$$n_x \;\to\; \sin\theta\cos\phi \quad , \quad n_y \;\to\; \sin\theta\sin\phi \quad , \quad n_z \;\to\; \cos\theta \quad , \tag{46}$$

and with $D_{zz} = D$,

$$\phi_{\rm Q} \;=\; \frac{D}{4r^3}\big(3\cos^2\theta - 1\big) \;=\; \frac{D}{2r^3} P_2(\cos\theta) \quad . \tag{47}$$

Here $P_2(z)$ is a Legendre polynomial.

• The series expansion can be continued most effectively using the generating function for Legendre polynomials. This is an appropriate path because the Coulomb potential corresponds to a central force, so that a polar coordinate description works best. This leads to treatments of electrostatics using **spherical harmonics** when considering complex distributions of charges.

[*Reading Assignment: Spherical harmonic structure of the general Coulomb potential: Section 41.*]

3.3 Dipoles in External Fields

Consider now a system of charges in an external electric field $\mathbf{E}(\mathbf{r}) = \nabla\phi(\mathbf{r})$. Suppose that the gradient of the field is small on the spatial scale λ of the charge ensemble, i.e. that $\lambda\,|\nabla\cdot\mathbf{E}| \ll 1$. Then one can expand the potential energy U of the charge ensemble in a perturbative manner using the values of the potential $\phi_0 = \phi(\mathbf{r}=\mathbf{0})$ and its derivatives at the origin:

L&L Sec. 42

$$U(\mathbf{r}) \;\equiv\; \sum_n q_n\,\phi(\mathbf{x}_n) \;=\; \phi_0 \sum_n q_n + (\nabla\phi)_0 \sum_n q_n\,\mathbf{x}_n + \ldots \quad . \tag{48}$$

The sums of the successive terms are just the total charge Q and the dipole moment $\mathbf{d}$, so writing $\mathbf{E}_0 = -(\nabla\phi)_0$ for the electric field at the origin, we have

$$U(\mathbf{r}) \;=\; Q\phi_0 - \mathbf{d}\cdot\mathbf{E}_0 + \frac{D_{ij}}{6}\,\frac{\partial^2\phi_0}{\partial x_i \partial x_j} + \ldots \quad . \tag{49}$$

The derivation of the quadrupole term is routine and is detailed in the book.

• Now let's assume that the external field is produced by another system of charges with dipole moment $\mathbf{d}_2$. Use subscript 1 to label the first system. Assume that both systems have zero net charge, so that $Q_1 = 0 = Q_2$. Then, the potential energy of ensemble 2 in the field of ensemble 1 is

$$U(\mathbf{r}) \;\approx\; -\mathbf{d}_2\cdot\mathbf{E}_1 \quad , \tag{50}$$

provided that both ensembles are sufficiently remote from each other, so that the approximately uniform field presumption applies. For a position vector $\mathbf{r} = r\mathbf{n}$ from the center of system 1 to that of system 2, we can employ the far-field result in Eq. (35) for a dipole's $\mathbf{E}$ field to yield

$$U(\mathbf{r}) \;\approx\; -\mathbf{d}_2\cdot\frac{3(\mathbf{n}\cdot\mathbf{d}_1)\,\mathbf{n} - \mathbf{d}_1}{r^3} \;=\; \frac{\mathbf{d}_1\cdot\mathbf{d}_2}{r^3} - \frac{3(\mathbf{d}_1\cdot\mathbf{r})\,(\mathbf{d}_2\cdot\mathbf{r})}{r^5} \quad . \tag{51}$$

From this one can deduce that for two dipoles in a side-by-side configuration of dipoles (with $\mathbf{d}_i\cdot\mathbf{r} \approx 0$; *sketch diagram*), the minimum energy is realized when their polarity is opposite, i.e. $\mathbf{d}_2 = -\alpha\mathbf{d}_1$ for $\alpha > 0$. Conversely, if the two dipoles are in a collinear configuration then the minimum energy is achieved for the same polarity, i.e. $\mathbf{d}_2 = \alpha\mathbf{d}_1$. These polarity properties extend to near-field configurations, and govern the mobility and viscosity of atoms and molecules in both inorganic and organic materials.