8. ELECTROSTATICS AND MAGNETOSTATICS

Matthew Baring — Lecture Notes for PHYS 532, Spring 2023

1 Coulomb's Law

With the complete set of electromagnetic field equations assembled, the task now is to assemble more details of how charges generate and respond to fields in configurations that approximate those commonly encountered. The starting point is **electrostatics**, where the electric field is time-independent and the magnetic field is zero. The two pertinent Maxwell's equations are **L&L Sec. 36**

$$\nabla \cdot \mathbf{E} \;=\; 4\pi\rho \quad \text{and} \quad \nabla \times \mathbf{E} \;=\; \mathbf{0} \quad . \tag{1}$$

The latter can be expressed in terms of the potential $\phi(\mathbf{r})$ using the definition $\mathbf{E} = -\nabla\phi$, and the former then takes the form of **Poisson's equation**:

$$\nabla^2\phi \;=\; -4\pi\rho \quad . \tag{2}$$

This can be solved for simple charge configurations using the mathematical technique of separation of variables. When there is no charge, the solutions are necessarily linear functions of coordinates, that if bounded at infinity, correspond to a trivial $\phi =$constant solution.

For a single point charge q at the origin, the simplest way to determine the field form is to use Gauss' theorem from vector analysis to convert a volume integration of Gauss' law to a surface integral. Thus

$$4\pi q \;=\; \int 4\pi\rho\, dV \;=\; \int \nabla \cdot \mathbf{E}\, dV \;=\; \int \mathbf{E} \cdot d\mathbf{\Sigma} \;=\; 4\pi E\, r^2 \tag{3}$$

since the potential must be spherically symmetric and the field therefore radial as it threads $\boldsymbol{\Sigma}$ isotropically. The magnitude and direction of the electric field are then described via **Coulomb's law**:

$$\mathbf{E} \;=\; \frac{q\hat{r}}{r^2} \quad\Rightarrow\quad \phi(\mathbf{r}) \;=\; \frac{q}{r} \quad . \tag{4}$$

The potential form is simply obtained using the gradient operator.

For a system of charges, this form can be superposed to obtain the contribution from the entire ensemble:

$$\phi \;=\; \sum_n \frac{q_n}{|\mathbf{r}-\mathbf{r}_n|} \quad\Rightarrow\quad \nabla^2\phi \;=\; -4\pi \sum_n q_n\, \delta^3(\mathbf{r}-\mathbf{r}_n) \quad . \tag{5}$$

We therefore can identify the Green's function $1/|\mathbf{r}-\mathbf{r}_n|$ for the Coulomb problem for any stationary charge configuration.

Motion in the Coulomb Field: The pedagogy will only cover charge motions in a static Coulomb potential through problems; this is the expedient path to learning. **L&L Sec. 39**

- For unbound charges interacting with a stationary test charge, the interaction leads to a scattering deflection of the mobile charge, whether the interaction is attractive or repulsive.

- For bound interactions, which are necessarily attractive (i.e., between charges of opposite sign), the relativistic nature of the interaction leads to unclosed orbits, in distinct contrast to the closed elliptical trajectories for the motion of non-relativistic charges in the Coulomb field.

1.1 Electrostatic Energies of Charges

The Coulomb field possesses an energy that is readily expressible via **L&L Sec. 37**

$$U_{\rm em} \; = \; \frac{1}{8\pi} \int E^2 \, dV \quad . \tag{6}$$

Since the potential scales as $1/r$, the electric field magnitude $E \sim 1/r^2$ and the integration down to zero radii is divergent. This poses a problem that underlines the limitations of a classical theory of electromagnetism on small scales. We can re-express the Coulomb field energy using $\mathbf{E} = -\nabla\phi$:

$$U_{\rm em} \; = \; -\frac{1}{8\pi} \int \mathbf{E} \cdot \nabla \phi \, dV \; = \; -\cancel{\frac{1}{8\pi} \int \nabla (\mathbf{E}\phi) \, dV} + \frac{1}{8\pi} \int \phi \, \nabla \cdot \mathbf{E} \, dV \quad . \tag{7}$$

This is essentially an integration by parts. The first integral on the right is of a pure derivative and is thus zero if the volume extends to infinity. The second integral on the right can be re-written using Gauss' law for electrostatics, $\nabla \cdot \mathbf{E} = 4\pi\rho$. The result is the compact form

$$U_{\rm em} \; = \; \frac{1}{2} \int \rho \, \phi \, dV \; \to \; \frac{1}{2} \sum_n q_n \phi(\mathbf{r}_n) \quad . \tag{8}$$

The second form here is the discrete charge ensemble distillation of the energy integral, with $\phi(\mathbf{r}_n)$ being the potential resulting from the ensemble at the point of charge q_n . For a single charge, the potential diverges at its position, and so one deduces an **infinite self-energy for electrostatics**.

- This divergence is untenable, and defines a limitation of classical electrodynamics. Such a self-energy could, in principle, *serve as an attribution for mass*. It certainly cannot exceed mc^2 . Thus, for an electron, the validity of classical electrodynamics must be restricted to radius domains such that

$$\frac{e^2}{r} \; \lesssim \; m_e c^2 \quad \Rightarrow \quad r \; \gtrsim \; r_0 \; \equiv \; \frac{e^2}{m_e c^2} \; \approx \; 2.818 \times 10^{-13}\,{\rm cm} \quad . \tag{9}$$

Accordingly, we define the fundamental limiting scale of classical E/M, namely the **classical electron radius** r_0 ; at comparable lengths, quantum mechanics must be introduced, arising naturally on scales $\sim \hbar / m_e c = r_0 / \alpha_{\rm f}$.

- Specifying a self-energy is therefore an approximate practice, and even in quantum mechanics, divergences appear and have to be eliminated through **renormalization techniques**.

1.2 Fourier Transform of Electrostatic Fields

As a preparation for wave elements of our electrodynamics pedagogy, it is instructive to consider the spatial Fourier transform of the Coulomb field. This is germane to structured static systems of charges, such as in a lattice. For single point charge at the origin, Poisson's equation assumes the form

L&L Sec. 51

$$\nabla^2\phi \;=\; -4\pi q\, \delta^3(\mathbf{r}) \quad . \tag{10}$$

Let us decompose the field in terms of its 3D space Fourier components $\phi_{\mathbf{k}}$ via the transform

$$\phi \;=\; \int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{r}}\, \phi_{\mathbf{k}}\, \frac{d^3k}{(2\pi)^3} \quad \Leftrightarrow \quad \phi_{\mathbf{k}} \;=\; \int e^{-i\mathbf{k}\cdot\mathbf{r}}\, \phi(\mathbf{r})\, d^3x \quad . \tag{11}$$

Here, $\mathbf{k}$ is the wavevector of the field transform, with $\lambda = 2\pi/|\mathbf{k}|$ being the effectively **wavelength** or spatial scale of **longitudinal** potential variations. Observe that this construction employs an asymmetric convention for the Fourier transform/inverse transform pair. Poisson's equation yields

$$\nabla^2\phi \;=\; \int_{-\infty}^{\infty} (ik)^2\, e^{i\mathbf{k}\cdot\mathbf{r}}\, \phi_{\mathbf{k}}\, \frac{d^3k}{(2\pi)^3} \;=\; -4\pi q\, \delta^3(\mathbf{r}) \;=\; -4\pi q \int e^{i\mathbf{k}\cdot\mathbf{r}}\, \frac{d^3k}{(2\pi)^3} \quad . \tag{12}$$

The last identity stems from the Fourier representation of the Dirac delta function. Equating integrands, it then follows that

$$\phi_{\mathbf{k}} \;=\; \frac{4\pi q}{k^2} \tag{13}$$

is the Fourier transform of the 3D Coulomb potential. If one extends this to a **screened Coulomb field** (say in a classical model of atom) with a potential $\phi(r) = q e^{-\mu r}/r$, then the Fourier transform becomes

$$\phi_{\mathbf{k}} \;=\; \frac{4\pi q}{k^2 + \mu^2} \quad . \tag{14}$$

Accordingly, $2\pi/\mu$ defines the lengthscale for screening of the bare Coulomb potential. This is pertinent to the Thomas-Fermi model of atoms, and **Debye screening** in warm plasmas.

• The electric field Fourier component $\mathbf{E_k}$ is simply ascertained by taking the gradient. Thus,

$$\begin{aligned} \mathbf{E} &\equiv \int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{r}}\, \mathbf{E_k}\, \frac{d^3k}{(2\pi)^3} \\ &= -\nabla \int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{r}}\, \phi_{\mathbf{k}}\, \frac{d^3k}{(2\pi)^3} \\ &= -i \int_{-\infty}^{\infty} \mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, \phi_{\mathbf{k}}\, \frac{d^3k}{(2\pi)^3} \quad . \end{aligned} \tag{15}$$

The Fourier transform of the electric field is thus

$$\mathbf{E_k} = -i\,\mathbf{k}\,\phi_{\mathbf{k}} = -i\,\frac{4\pi q\,\mathbf{k}}{k^2} \quad . \tag{16}$$

2 Field of a Charge in Uniform Motion

To prepare for the radiation portions of the course, we need to characterize fields of charges in uniform motion. Let the rest frame of a charge be K', moving with a velocity $\mathbf{v} = v\hat{x} = \beta c\hat{x}$ with respect to our K frame. **L&L Sec. 38**

Plot: K frame and frame K' in which charge is at rest.

The scalar potential is easily transformed between the two frames, since the vector potential in the charge rest frame is $\mathbf{A}' = 0$. Thus,

$$\phi = \frac{\phi'}{\sqrt{1-\beta^2}} = \frac{q}{r'\sqrt{1-\beta^2}} \quad \text{for} \quad (r')^2 = (x')^2 + (y')^2 + (z')^2 \quad . \tag{17}$$

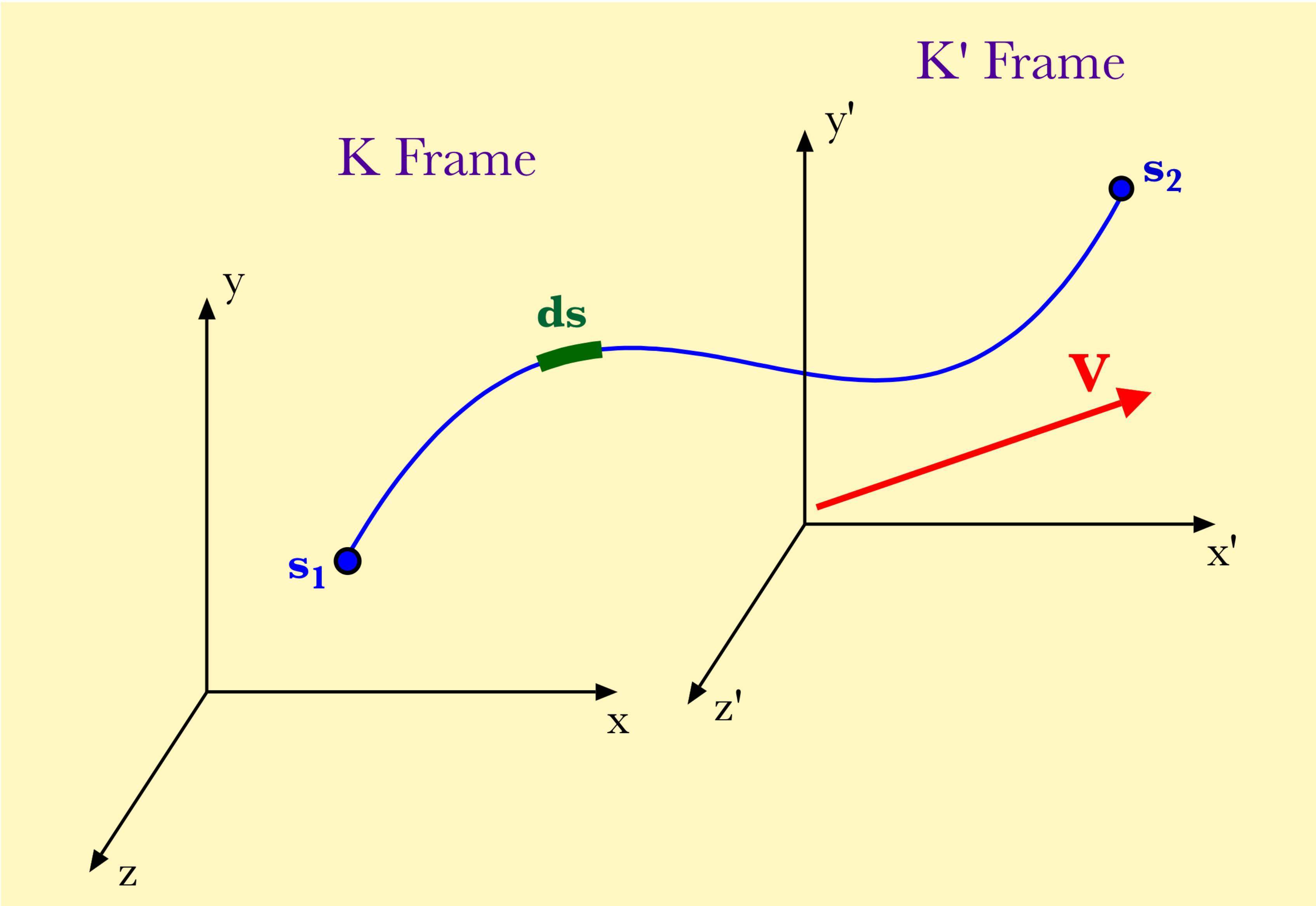
These K' frame coordinates need to be expressed in terms of our K frame coordinates $(t,\, x,\, y,\, z)$. This is effected using the Lorentz boost relations:

$$x' = \gamma(x - \beta ct) \quad , \quad y' = y \quad , \quad z' = z \quad . \tag{18}$$

It follows then that

$$(r')^2 = \gamma^2(x-\beta ct)^2 + y^2 + z^2 \quad , \tag{19}$$

Two Inertial Frames in Relative Motion



which can be substituted directly into the Coulomb form. The vector potential in the K frame can be similarly prescribed using $\mathbf{A} = \gamma\phi'\mathbf{v}/c = \phi\mathbf{v}/c$, since $\mathbf{A}' = 0$. If one then sets

$$\left(r^*\right)^2 \;\equiv\; \frac{(r')^2}{\gamma^2} \;=\; (x-\beta ct)^2 + \left(1-\beta^2\right)\left(y^2+z^2\right) \quad , \tag{20}$$

then the components of the four-potential in the K frame are

$$\phi \;=\; \frac{q}{r^*} \quad , \quad \mathbf{A} \;=\; \frac{q\,\mathbf{v}}{cr^*} \;=\; \frac{q\beta}{r^*}\,\hat{x} \quad . \tag{21}$$

The electric field can be obtained in either of two ways. One path is to use the Coulomb field form in the charge's rest frame and Lorentz transform it using our standard boost relations. This gives

$$E_x \;=\; E_x' \;=\; \frac{qx'}{(r')^3} \quad , \quad E_y \;=\; \gamma E_y' \;=\; \frac{\gamma q y'}{(r')^3} \quad , \quad E_z \;=\; \frac{\gamma q z'}{(r')^3} \;. \tag{22}$$

This is the simplest approach, yet the same equations will result by taking the 4-gradient of the 4-potential $(\phi, \mathbf{A})$ using the expression for r' in Eq. (19). The field can then be written in vector notation:

$$\mathbf{E} \;\equiv\; -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \;=\; \frac{q\,\mathbf{r}}{\gamma^2(r^*)^3} \quad \text{for} \quad \mathbf{r} \;=\; (x-\beta ct,\, y, z) \quad . \tag{23}$$

This is the first indication that there is a **retardation** or time delay in establishing the field: causality applies in communicating electromagnetic information from a moving charge.