## 3 The Particle Energy-Momentum Tensor

**L&L Sec. 35**

The next piece of the puzzle is to add in particles to the formalism. The matter can be presumed to be non-interacting at first. It can be described by a **mass density function** in a center-of-momentum frame

$$\mu = \sum_n m_n \delta^3(\mathbf{r} - \mathbf{r}_n) \quad , \tag{42}$$

which is a mass analog of the charge density function that was employed when defining the four-current vector. Therefore the total rest mass is

$$m = \int \mu(\mathbf{r})\, dV \quad . \tag{43}$$

As such masses move along world lines, their total energy and momentum must be conserved, and so we need an energy-momentum tensor to express this. The matter portion of the action is then (with $\gamma \to 1$ in CM frame)

$$S_m = -\int mc\, ds = -\int \mu(x^\nu) c\, dV\, ds = -\int \frac{\mu c}{\gamma}\, d^4x \quad . \tag{44}$$

We thus can identify the **Lagrangian density** for the matter:

$$\Lambda_m = \frac{\mu c^2}{\gamma} \sqrt{u_\nu u^\nu} \quad . \tag{45}$$

The last factor (nominally unity), is introduced to render it consistent with the covariant Lagrangian formulation in Chapter 4. The energy-momentum tensor can then be derived using

$$T_\mu^\nu = \dot{q}_\mu \frac{\partial \Lambda}{\partial \dot{q}_\nu} - \delta_\mu^\nu \Lambda \quad . \tag{46}$$

The coordinates here are obviously $q_\nu \to x_\mu$ so that the their derivatives are the four-velocity $\dot{q}_\nu \to u_\nu$ . As $\gamma$ is constant for unforced motion (no fields),

$$\frac{\partial \Lambda}{\partial \dot{q}_\nu} = \frac{1}{2}\, 2\, \frac{\mu c^2}{\gamma}\, u^\nu \quad . \tag{47}$$

It follows that the **matter energy-momentum tensor** takes the form

$$T^{\mu\nu} = \frac{\mu c^2}{\gamma} \left( u^\mu u^\nu - \eta^{\mu\nu} \cancel{\sqrt{u_\alpha u^\alpha}} \right) \to \frac{\mu c^2}{\gamma}\, u^\mu u^\nu \quad . \tag{48}$$

The second term will be neglected for now, since it is a constant rest mass energy term, upon which the energy-momentum conservation law does not depend. With or without it, the tensor is symmetric.

• Since $T^{00} = \gamma\mu c^2$ as $u^\mu = \gamma(1, \mathbf{v})$, it defines the *energy density.* For the time-space components, they are $T^{0i} = \gamma\mu v^i c$ and so define fluxes of relativistic momentum density, again as they should. The other components are in accord with our original interpretation of the energy-momentum tensor. We note that the diagonal space-space elements are $T^{ii} = \gamma\mu v^i v^i$.

From the elemental description, one can write down an alternative form for the tensor:

$$T^{\mu\nu} \;=\; \sum_n \frac{m_n c^2}{\gamma_n}\, u_n^\mu u_n^\nu\, \delta^3(\mathbf{r}-\mathbf{r}_n) \;=\; \sum_n \frac{p_n^\mu p_n^\nu c^2}{E_n}\, \delta^3(\mathbf{r}-\mathbf{r}_n) \quad . \tag{49}$$

This also displays the symmetrical nature of the tensor.

Now let's consider a system that is at rest, but that contains complex internal motions for all its particles. In other words, *it is stationary, but contains heat.* Since the system has zero net momentum, the $T^{0i}$ components will be identically zero. So also will all the other off-diagonal components. The $T^{00}$ element is the energy density $\mathcal{E}$. The diagonal space-space elements describe the flux of momentum components across surfaces locally perpendicular to the particular direction: $T^{11}$ defines the flux in the $x$-direction of the $x$-component of momentum. Therefore,

$$T^{ii} \;=\; \frac{\text{momentum}}{\text{volume}} \boxtimes \frac{\text{distance}}{\text{time}} \;=\; \frac{\text{force}}{\text{area}} \quad . \tag{50}$$

Accordingly, these define pressures, and if the matter is isotropic, then $-\sigma^{ii} = P$, i.e. $T^{11} = T^{22} = T^{33} = P$. The matter energy-momentum tensor is

$$T^{\mu\nu} \;=\; \begin{pmatrix} \mathcal{E} & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \quad . \tag{51}$$

This is a form that is of wide applicability: to fluids, gases and plasmas, and macroscopic bodies. It is an important construct for general relativity. In terms of our elemental description, the energy and pressure can be written:

$$\mathcal{E} \;=\; \sum_n E_n\, \delta^3(\mathbf{r}-\mathbf{r}_n) \quad , \quad P \;=\; \sum_n \frac{p_n^j v_n^j c}{3}\, \delta^3(\mathbf{r}-\mathbf{r}_n) \quad . \tag{52}$$

Thus, the ensemble averages for the energy density and pressure are

$$\mathcal{E} \;=\; T^{00} \;=\; nmc^2 \,\langle\gamma\rangle \quad , \quad P \;=\; \frac{T^{ii}}{3} \;=\; \frac{nmc^2}{3}\,\langle\gamma\beta^2\rangle \quad , \tag{53}$$

where $n$ is the number density. Thus $nm = \mu$.

**Plot:** Draw a wall with momentum impact as a model of pressure.

• Not always is the system isotropic. For example, external magnetic fields and turbulence can yield anisotropic momentum transport, thereby modifying both the diagonal and off-diagonal elements.

A relativistic boost of an isotropic system *will inherently render the distribution of masses anisotropic.* Accordingly, one anticipates that the off-diagonal elements of $T^{\mu\nu}$ will be populated. Consider a boost $\beta\hat{x}$. Then

$$T^{\mu\nu} \;=\; \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathcal{E} & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \cdot \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad . \tag{54}$$

The matrix algebra generates

$$T^{\mu\nu} \;=\; \begin{pmatrix} \gamma^2(\mathcal{E}+\beta^2 P) & \gamma^2\beta(\mathcal{E}+P) & 0 & 0 \\ \gamma^2\beta(\mathcal{E}+P) & \gamma^2(P+\beta^2\mathcal{E}) & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \quad . \tag{55}$$

Remembering that $u^\nu = \gamma(1,\beta,0,0)$ in this case, we procure the form

$$\begin{aligned} T^{\mu\nu} \;&=\; (\mathcal{E}+P)\, u^\mu u^\nu - P\,\eta^{\mu\nu} \quad , \\ \text{or}\quad T^\mu_\nu \;&=\; (\mathcal{E}+P)\, u^\mu u_\nu - P\,\delta^\mu_\nu \quad . \end{aligned} \tag{56}$$

This is the general form for the **energy-momentum tensor for matter**, applicable to arbitrary boost directions In the case of zero pressure, i.e. no internal motions, it reduces to our original form, $T^{\mu\nu} = \mu c^2\, u^\mu u^\nu$.

• Note that the Lorentz boost mixes pressure and energy contributions to the new energy and momentum densities. Thus relativistic motion converts internal energy to bulk kinetic energy and *vice versa*.

• The coexistence of pressure and energy density in $T^{\mu\mu}$ within the construct of relativity implies that *internal heat energy contributes to the mass of a system.* This is true for macroscopic bodies and also for quantum systems.

• Also, the Lorentz transformation protocol required the presence of rest mass energy in the original stress-energy tensor. This motivated our reasoning for not subtracting it out when working from the Lagrangian density.

The energy density $U_m$ and energy flux vector $\mathbf{S}$ are obvious:

$$U_m \;=\; \gamma^2(\mathcal{E} + \beta^2 P) \quad , \quad \mathbf{S} \;=\; \gamma^2(\mathcal{E} + P)\mathbf{v} \quad . \tag{57}$$

From this we determine that since $0 < \mathcal{P} < \mathcal{E}$,

$$U_m^2 - \frac{\mathbf{S}^2}{c^2} \;=\; \gamma^2\Big(\mathcal{E}^2 - \beta^2 P^2\Big) \;> 0 \quad . \tag{58}$$

This distinctly contrasts the electromagnetic field case where this quantity is zero. This is the signature of massive matter that travels at less than $c$.

The trace of the matter energy-momentum tensor is simply obtained:

$$T^{\mu}_{\mu} \;=\; (\mathcal{E} + P)\, u^{\mu} u_{\mu} - P\, \delta^{\mu}_{\mu} \;=\; \mathcal{E} - 3P \quad . \tag{59}$$

This form is exactly that obtained for the system at rest, as expected since Lorentz boosts do not alter the traces of tensors. Because the rest-system form compares pressures to rest mass energies, the sum of the squares of the velocity components is always inferior to the total energy, rest mass plus kinetic energy, per unit mass. This can be seen from the form

$$T^{\mu}_{\mu} \;=\; \sum_n \frac{p_n^{\mu} p_{n\mu} c^2}{E_n}\, \delta^3(\mathbf{r} - \mathbf{r}_n) \;=\; \sum_n m_n c^2 \sqrt{1 - \frac{v_n^2}{c^2}}\, \delta^3(\mathbf{r} - \mathbf{r}_n) \quad , \tag{60}$$

which is always positive definite. Therefore the trace must be positive, and

$$P \;<\; \frac{\mathcal{E}}{3} \quad , \tag{61}$$

a constraint on the **equation of state** of the system. Only when rest mass is negligible and the mean speed of particles is very close to $c$ can $P \approx \mathcal{E}/3$ conditions be realized, and the equation of state is ultra-relativistic.

• The mass density function can often obey its own conservation law. This is not sacrosanct in that mass does not always have to be conserved. But when it is, it can be cast as a divergence of a four-vector:

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} \;=\; 0 \quad \Rightarrow \quad \frac{\partial}{\partial x^\nu}\left(\frac{\mu\, u^\nu}{\gamma}\right) \;=\; 0 \quad , \tag{62}$$

where $P \to 0$ is set because we are considering only mass, and the two four-velocity factors in $T^{\mu\nu}$ introduce a factor of two, not explicitly shown.

[*Reading Assignment: Last part of L&L Section 33: Proof using the $T^{\mu\nu}$ tensor that the combined E/M and matter system conserves four-momentum.*]

## 3.1 The Virial Theorem for Ensembles of Particles

A fundamental energy theorem for closed systems of isolated particles can be derived from the energy-momentum conservation law. We have **L&L Sec. 34**

$$\frac{\partial T^\nu_\mu}{\partial x^\nu} \;\equiv\; \frac{1}{c}\,\frac{\partial T^0_\mu}{\partial t} + \frac{\partial T^i_\mu}{\partial x^i} \;=\; 0 \quad . \tag{63}$$

Now take a long-term time average, denoted by $\langle \dots \rangle_t$ . *If the time interval is large enough, the time variation must average to zero*:

$$\left\langle \frac{\partial T^0_\mu}{\partial t} \right\rangle_t \;\equiv\; \frac{1}{T}\int_0^T \frac{\partial T^0_\mu}{\partial t}\, dt \;=\; \frac{T^0_\mu(T) - T^0_\mu(0)}{T} \;\to\; 0 \quad , \tag{64}$$

as $T \to \infty$ . Thus, one concludes that the space component satisfies

$$\left\langle \frac{\partial T^i_\mu}{\partial x^i} \right\rangle_t \;=\; 0 \quad , \quad \mu \;=\; 0, 1, 2, 3 \quad . \tag{65}$$

Now consider only the space components, $\mu \to j$ . This can be weighted by the space vector components, and integrated over a space volume that extends to infinity, the result obviously being zero by virtue of Eq. (65). This is tantamount to exploring rates of change of the energy, i.e. dot products between force and velocity, and this protocol forms a **virial**. Thus,

$$0 \;=\; \int x^j \left\langle \frac{\partial T^i_j}{\partial x^i} \right\rangle_t dV \;=\; -\int \frac{\partial x^j}{\partial x^i} \left\langle T^i_j \right\rangle_t dV \;=\; -\int \delta^j_i \left\langle T^i_j \right\rangle_t dV \quad . \tag{66}$$

The integration by parts includes the integral of a perfect derivative that is identically zero as no matter exists at infinity. Therefore, we have evaluated the time average of the space components of the trace of the energy-momentum tensor $T^{\mu}_{\nu}$ :

$$\int \left\langle T^{j}_{j} \right\rangle_t dV \; = \; 0 \quad . \tag{67}$$

We conclude that the total trace of the tensor satisfies

$$\int \left\langle T^{\mu}_{\mu} \right\rangle_t dV \; = \; \int \left\langle T^{0}_{0} \right\rangle_t dV \; \equiv \; \mathcal{U} \; \equiv \; \sum_n m_n c^2 \left\langle \sqrt{1 - \frac{v_n^2}{c^2}} \; \right\rangle_t \quad , \tag{68}$$

i.e. the total energy of the system. The last equivalence follows from Eq. (60). This relation is the **relativistic virial theorem**, an extension from classical mechanics. If we subtract off the rest mass energy, the result is a negative total energy, kinetic plus potential, to the system:

$$\mathcal{U} - \mathcal{M}c^2 \; = \; \sum_n m_n c^2 \left\langle \left\{ \sqrt{1 - \frac{v_n^2}{c^2}} \; - 1 \right\} \right\rangle_t \; < \; 0 \quad . \tag{69}$$

This reflects the fact that for no mass or energy to be present at infinity requires the system to be bound.