2 The Electromagnetic $T^{\mu\nu}$ Tensor

The Lagrangian density and action for the electromagnetic field are **L&L Sec. 33**

$$\Lambda_{\rm em} \;=\; -\frac{1}{16\pi}\,F_{\mu\nu}F^{\mu\nu} \quad \text{with} \quad S_{\rm em} \;=\; \int \Lambda_{\rm em}\,d^4x \quad . \tag{22}$$

The "coordinates" q in the covariant formulation of the equations of motion are the components of the 4-potential A^μ. It then follows that the energy-momentum tensor for the field is

$$T_\mu^\nu \;=\; \dot{q}_\mu\,\frac{\partial\Lambda}{\partial\dot{q}_\nu} - \delta_\mu^\nu\Lambda \;\to\; \frac{\partial A_\lambda}{\partial x^\mu}\,\frac{\partial\Lambda}{\partial[\partial A_\lambda/\partial x^\nu]} - \delta_\mu^\nu\Lambda \quad . \tag{23}$$

The gradients of the 4-potential, are, of course, the components of the electric and magnetic fields, and so these should appear explicitly in the energy-momentum tensor. Remember that the field tensor is

$$F_{\alpha\beta} \;=\; \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} \quad . \tag{24}$$

It follows that

$$\frac{\partial\Lambda}{\partial[\partial A_\lambda/\partial x^\nu]} \;=\; -\frac{1}{8\pi}F^{\alpha\beta}\,\frac{\partial F_{\alpha\beta}}{\partial[\partial A_\lambda/\partial x^\nu]} \;=\; -\frac{1}{8\pi}F^{\alpha\beta}\Big(\delta_\beta^\lambda\delta_\alpha^\nu - \delta_\alpha^\lambda\delta_\beta^\nu\Big)\,. \tag{25}$$

Then we employ the anti-symmetry $F^{\beta\alpha} = -F^{\alpha\beta}$ and relabel the second term on the right $\alpha \leftrightarrow \beta$. Thus,

$$\frac{\partial\Lambda}{\partial[\partial A_\lambda/\partial x^\nu]} \;=\; -\frac{1}{4\pi}F^{\alpha\beta}\delta_\beta^\lambda\delta_\alpha^\nu \;=\; -\frac{1}{4\pi}F^{\nu\lambda} \quad . \tag{26}$$

Gathering together the results, we have

$$T_\mu^\nu \;=\; -\frac{1}{4\pi}\,\frac{\partial A_\lambda}{\partial x^\mu}\,F^{\nu\lambda} + \frac{1}{16\pi}\,\delta_\mu^\nu\,F_{\alpha\beta}F^{\alpha\beta} \quad , \tag{27}$$

or in contravariant form,

$$T^{\mu\nu} \;=\; -\frac{1}{4\pi}\,\frac{\partial A^\lambda}{\partial x_\mu}\,F_\lambda^\nu + \frac{1}{16\pi}\,g^{\mu\nu}\,F_{\alpha\beta}F^{\alpha\beta} \quad . \tag{28}$$

Remember that in raising indices we are employing the Minkowski metric $g^{\mu\nu} = \eta^{\mu\nu}$. Moreover, now we see why the mixed derivatives of A^λ were employed throughout this derivation: they generate the simpler δ tensor.

The tensor so derived is not symmetric. To render it thus, we add a term

$$\frac{1}{4\pi}\,\frac{\partial A^{\mu}}{\partial x_{\lambda}}\,F^{\nu}_{\lambda} \;=\; \frac{1}{4\pi}\,\frac{\partial}{\partial x_{\lambda}}\Big(A^{\mu}F^{\nu}_{\lambda}\Big) - \frac{A^{\mu}}{4\pi}\,\cancel{\frac{\partial F^{\nu}_{\lambda}}{\partial x_{\lambda}}} \quad . \tag{29}$$

The gradient factor in the second term is just that which appears in the covariant form of the two Maxwell equations with source contributions, i.e. is proportional to the four-current j^{μ}. Since we are presuming an absence of charges, this is identically zero. Hence, the added term is a perfect derivative, the gradient of a function that is anti-symmetric in the last two indices by virtue of the F^{ν}_{λ}. Therefore, this addition *is a permissible gauge transformation.* Accordingly, another electromagnetic field tensor appears.

The final form of the **electromagnetic energy-momentum tensor** is

$$T^{\mu\nu} \;=\; \frac{1}{4\pi}\left(-F^{\mu\lambda}F^{\nu}_{\lambda} + \frac{1}{4}\,g^{\mu\nu}\,F_{\alpha\beta}F^{\alpha\beta}\right) \tag{30}$$

It is clearly symmetric under $\mu \leftrightarrow \nu$:

$$4\pi\Big(T^{\mu\nu} - T^{\nu\mu}\Big) \;=\; -F^{\mu\lambda}F^{\nu}_{\lambda} + F^{\nu\lambda}F^{\mu}_{\lambda} \;\equiv\; -F^{\mu\lambda}F^{\nu}_{\lambda} + F^{\nu}_{\lambda}F^{\mu\lambda} \;=\; 0 \quad . \tag{31}$$

The identification of the components of the electromagnetic energy-momentum tensor is now routine. For the first part, we use first

$$F^{\nu}_{\lambda} \;=\; F^{\nu\beta}\eta_{\beta\lambda} \;=\; \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}^{T} \eta_{\beta\lambda} \;=\; \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \;,$$

where transposition (T) is needed for summing over second index (β). Then,

$$F^{\mu\lambda}F^{\nu}_{\lambda} \;=\; \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{32}$$

Accordingly, the first part of the stress-energy tensor is

$$-F^{\mu\lambda}F^{\nu}_{\lambda} \;=\; \begin{pmatrix} E_x^2 + E_y^2 + E_z^2 & B_zE_y - B_yE_z & -B_zE_x + B_xE_z & B_yE_x - B_xE_y \\ B_zE_y - B_yE_z & B_y^2 + B_z^2 - E_x^2 & -B_xB_y - E_xE_y & -B_xB_z - E_xE_z \\ -B_zE_x + B_xE_z & -B_xB_y - E_xE_y & B_x^2 + B_z^2 - E_y^2 & -B_yB_z - E_yE_z \\ B_yE_x - B_xE_y & -B_xB_z - E_xE_z & -B_yB_z - E_yE_z & B_x^2 + B_y^2 - E_z^2 \end{pmatrix}$$

The second part just involves one of the electromagnetic field invariants:

$$\frac{1}{4}\, g^{\mu\nu}\, F_{\alpha\beta}F^{\alpha\beta} \;=\; \frac{1}{2}\Bigl(\mathbf{B}^2 - \mathbf{E}^2\Bigr)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad . \qquad (33)$$

Adding the two together gives familiar results for the components involving time:

$$T^{00} \;=\; U_{\rm em} \;=\; \frac{E^2 + B^2}{8\pi} \quad , \quad T^{0i} \;=\; \frac{1}{c}\, S_i \;=\; \frac{(\mathbf{E}\times\mathbf{B})_i}{4\pi} \quad . \qquad (34)$$

The time-time component is just the energy density of the E/M field. The time-space components $P^i c$ thus involve c times the Poynting vector: from this, we can immediately infer that *the Poynting vector represents the momentum density of the electromagnetic field*.

The space-space diagonal components assume forms like

$$T^{11} \;=\; -\sigma_{xx} \;=\; \frac{1}{8\pi}\Bigl(\mathbf{E}^2 + \mathbf{B}^2 - 2E_x^2 - 2B_x^2\Bigr) \;=\; U_{\rm em} - \frac{E_x^2 + B_x^2}{4\pi} \quad . \qquad (35)$$

Summing these diagonal components gives the **trace** of the $T^{\mu\nu}$ tensor:

$$T^{\nu}_{\nu} \;=\; T^0_0 - T^1_1 - T^2_2 - T^3_3 \;=\; T^{00} - T^{11} - T^{22} - T^{33} \;=\; 0 \quad . \qquad (36)$$

The traceless nature of the electromagnetic energy-momentum tensor is an important property. It distinguishes from the case of the matter $T^{\mu\nu}$, which, as we shall soon see, has a non-zero trace marking the mass of the matter. From this character, we infer that the **electromagnetic field is massless**. The remaining off-diagonal space-space components are of forms like

$$T^{12} \;=\; -\sigma_{xy} \;=\; -\frac{1}{4\pi}\Bigl(E_xE_y + B_xB_y\Bigr) \quad . \qquad (37)$$

The space-space components can then be expressed in a compact form

$$\sigma_{ij} \;=\; \frac{E_iE_j + B_iB_j}{4\pi} - \delta_{ij}U_{\rm em} \quad , \quad i,j \;=\; 1,2,3 \quad , \qquad (38)$$

which is a 3-tensor known as the **Maxwell stress tensor**.

Gathering together all results, the <u>symmetric</u> energy-momentum tensor is

$$T^{\mu\nu} \;=\; \begin{pmatrix} U_{\rm em} & S_x/c & S_y/c & S_z/c \\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{zx} \\ S_y/c & -\sigma_{xy} & -\sigma_{yy} & -\sigma_{yz} \\ S_z/c & -\sigma_{zx} & -\sigma_{yz} & -\sigma_{zz} \end{pmatrix} \quad . \tag{39}$$

Generally this is a complicated form. However, if **E** and **B** are not mutually perpendicular and equal in magnitude, we can always find a Lorentz boost that will render the two fields parallel to each other. In such a frame, the Poynting vector is zero, and the off-diagonal elements of the Maxwell stress tensor are also zero. In this special frame, with $\mathbf{E} \parallel \mathbf{B}$, the $T^{\mu\nu}$ tensor is diagonal; if the direction of the fields is along the x-axis, then

$$T^{00} \;=\; -T^{11} \;=\; T^{22} \;=\; T^{33} \;=\; U_{\rm em} \quad . \tag{40}$$

This diagonalization procedure with 4D rotations is possible because $T^{\mu\nu}$ is a **Hermitian** tensor.

- The case where **E** and **B** <u>are</u> mutually perpendicular and equal in magnitude is of key interest. Let us suppose that **E** is in x-direction, and **B** is in the y-direction. The off-diagonal σ_{ij} are identically zero, as are σ_{xx} and σ_{yy}. Furthermore, the Poynting vector components in the x and y-directions are also zero. Given that the energy density is $U_{\rm em} = E^2/4\pi = B^2/4\pi$ in this special case,

$$T^{\mu\nu} \;=\; \begin{pmatrix} U_{\rm em} & 0 & 0 & U_{\rm em} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ U_{\rm em} & 0 & 0 & U_{\rm em} \end{pmatrix} \quad . \tag{41}$$

since $-\sigma_{zz} = U_{\rm em} = S_z/c$. Thus the flux of momentum/energy is purely along a direction perpendicular to the fields. The energy flow is at speed c, since $T^{03}/T^{00} = 1$, and thus electromagnetic signals (light) must involve transverse fields, i.e tranverse to **S**.

∗ In contrast to the other more general case, $T^{\mu\nu}$ is not diagonalizable in this special case, since its determinant is zero. The reason is that it describes light, which is inherently a transverse wave. The required boost would need to be of speed c, which is physically inadmissable.