7. THE ENERGY-MOMENTUM TENSOR

Matthew Baring — Lecture Notes for PHYS 532, Spring 2023

1 The Energy-Momentum Tensor $T^{\mu\nu}$

L&L Sec. 32

Now that we have identified energy density and momentum for the electromagnetic field, we need to establish a tensor that captures such information, and provides the opportunity for the analogous quantities for matter/particles to be incorporated on a similar footing under a single combined tensor. Accordingly, we return to the action, which has both matter and field contributions. Let us re-write the action:

$$S \;=\; \int L\,dt \;=\; \int \Lambda\left(q,\,\frac{\partial q}{\partial x^\mu}\right)\,dV\,dt \;\equiv\; \frac{1}{c}\int \Lambda\,d^4x \quad . \tag{1}$$

Here $d^4x = dV\,dt$ is the covariant space-time volume element. Thus, Λ is a covariant quantity which has the dimensions of the Lagrangian divided by the volume; thus we refer to it as the **Lagrangian density**. While one coordinate q is specified here (defining the state of the system), it is understood that this can be extended to numerous q_i coordinates.

The equations of motion are again defined by the variational principle. For compactness of notation, we write $\dot{q}_\mu \equiv \partial q/\partial x^\mu$ in what follows. Thus

$$\begin{aligned}
\delta S \;&=\; \frac{1}{c}\int\left(\frac{\partial\Lambda}{\partial q}\,\delta q + \frac{\partial\Lambda}{\partial\dot{q}_\mu}\,\delta\dot{q}_\mu\right)d^4x \\
&=\; \frac{1}{c}\int\left(\frac{\partial\Lambda}{\partial q}\,\delta q + \cancel{\frac{\partial}{\partial x^\mu}\left[\frac{\partial\Lambda}{\partial\dot{q}_\mu}\,\delta q\right]} - \delta q\,\frac{\partial}{\partial x^\mu}\left[\frac{\partial\Lambda}{\partial\dot{q}_\mu}\right]\right)d^4x \;=\; 0 \quad .
\end{aligned} \tag{2}$$

On the second line, the middle term can be expanded as the sum of the last term on the second line, and the second on the first line via the chain rule of differentiation. This same second term is a perfect derivative, and so it vanishes upon integration over d^4x once the "endpoints" are fixed. We are thus left with another form for the **covariant Euler-Lagrange equations**:

$$\boxed{\frac{\partial}{\partial x^\mu}\left[\frac{\partial \Lambda}{\partial \dot{q}_\mu}\right] - \frac{\partial \Lambda}{\partial q} \;=\; 0 \quad .} \tag{3}$$

While this expresses the motion in just one co-ordinate, we can easily extend to full spacetime via four-vector constructions. This we generalize to $q \to x^\nu$ for particles, and $q \to A^\nu$ for fields. It then follows that we have 8 equations of motion in a fully covariant electromagnetic/charge system. These, of course are constituted by the Newton-Lorentz force equation (4 components) and Maxwell's inhomogenous equations, namely Gauss' Law of electrostatics and Ampere's Law (another 4 equations).

$*$ Note that the two homogeneous Maxwell's equations (4 components) are obtained as derivatives of the Lorentz force construction and are not independent, essentially being coupled under the $S_{\rm m} - S_{\rm mf}$ umbrella through the conservation of 4-momentum of moving charges/masses.

• This extended formulation now captures information on gradients of energy and momentum evolution in different directions in spacetime. This provides an encompassing structure *for describing conservation of energy, momentum and angular momentum* in generalized systems with evolving motions and electromagnetic fields.

$*$ This is actually of fundamental importance because changes in energy/momentum configurations of particle at one spacetime position are coupled to fields and particle motions in remote spacetime locations, and the causal conservation laws constitute a useful mathematical prescription.

To define the conservation laws, we form spacetime gradients of the Lagrangian density:

$$\frac{\partial \Lambda}{\partial x^\mu} \;=\; \frac{\partial \Lambda}{\partial q}\,\frac{\partial q}{\partial x^\mu} + \frac{\partial \Lambda}{\partial \dot{q}_\nu}\,\frac{\partial \dot{q}_\nu}{\partial x^\mu} \quad . \tag{4}$$

We next substitute for the derivative of Λ in the first term using the Euler-

Lagrange equations:

$$\frac{\partial \Lambda}{\partial x^{\mu}} \;=\; \dot{q}_{\mu}\,\frac{\partial}{\partial x^{\nu}}\left[\frac{\partial \Lambda}{\partial \dot{q}_{\nu}}\right] + \frac{\partial \Lambda}{\partial \dot{q}_{\nu}}\,\frac{\partial \dot{q}_{\nu}}{\partial x^{\mu}} \;\equiv\; \frac{\partial}{\partial x^{\nu}}\left(\dot{q}_{\mu}\,\frac{\partial \Lambda}{\partial \dot{q}_{\nu}}\right) \quad . \tag{5}$$

The last step requires noticing the equivalence of second-order derivatives:

$$\frac{\partial \dot{q}_{\nu}}{\partial x^{\mu}} \;\equiv\; \frac{\partial^2 q}{\partial x^{\mu}\partial x^{\nu}} \;\equiv\; \frac{\partial \dot{q}_{\mu}}{\partial x^{\nu}} \quad . \tag{6}$$

To clean this gradient identity up, we recast the derivative on the left using the identity tensor δ^{ν}_{μ} thus:

$$\frac{\partial \Lambda}{\partial x^{\mu}} \;=\; \delta^{\nu}_{\mu}\,\frac{\partial \Lambda}{\partial x^{\nu}} \quad . \tag{7}$$

This then leads to the definition of a new tensor quantity

$$T^{\nu}_{\mu} \;=\; \dot{q}_{\mu}\,\frac{\partial \Lambda}{\partial \dot{q}_{\nu}} - \delta^{\nu}_{\mu}\Lambda \quad . \tag{8}$$

This possesses units of energy density, and the mathematical form of a *covariant Hamiltonian*. It is termed the **energy-momentum tensor**, and is sometimes referred to as the **stress-energy tensor**. With this definition, the gradient of the Lagrangian density takes the form

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} \;=\; 0 \quad . \tag{9}$$

Given the dimensional nature of the energy-momentum tensor, this relation must be interpreted as **the conservation of 4-momentum**. It applies to both charges and fields.

- Again, the construction of the energy-momentum tensor can readily be extended to a multitude of coordinates q_{μ} and canonical momenta $\dot{q}_{\mu}$.

- By formulating this fundamental principle in this way, it captures the law for ensembles of particles and extended field configurations. The charges can all travel different world lines that may be causally connected via the electromagnetic fields they both generate and respond to. The entire matter/field ensemble will be subject to 4-momentum conservation.

1.1 Four-momentum of a Matter-Field Ensemble

Now we draw upon our study of charge conservation. Therein, we found that the vanishing of a four-gradient of a four-vector, the four-current, corresponds to conservation of the total charge. Here we have the gradient of a four-tensor, and so its conservation entails the conservation of four quantities. Let us define them to be

$$P^\mu \equiv \frac{1}{c}\int_\Sigma T^{\mu\nu} d\Sigma_\nu \quad . \tag{10}$$

Here $d\Sigma_\nu$ represents a surface normal element applicable to any 3D hypersurface, which could be a 3D space volume, or not. The normalization constant could be arbitrary, but the choice of $1/c$ will soon be motivated. The conservation law can then be used to express

$$0 \;=\; \int \frac{\partial T^{\mu\nu}}{\partial x^\nu}\, d^4x \;=\; \oint_\Sigma T^{\mu\nu} d\Sigma_\nu \quad . \tag{11}$$

In this case, we have invoked the generalization of Gauss' law to larger dimensions, so that here the hypersurface Σ is closed.

Plot: Hand draw connected hypersurfaces that combined are closed.

Now divide this closed hypersurface into two pieces, Σ_1 and Σ_2. This identity therefore expresses the invariance

$$P^\mu\Big|_1 \equiv \frac{1}{c}\int_{\Sigma_1} T^{\mu\nu} d\Sigma_\nu \;=\; -\frac{1}{c}\int_{\Sigma_2} T^{\mu\nu} d\Sigma_\nu \;\equiv\; -P^\mu\Big|_2 \quad . \tag{12}$$

Accordingly, the quantity P^μ is conserved. Choose one of these open hypersurfaces to be an *infinite hyperplane* $x^0 = \text{const}$. This is an infinite volume at constant time, and is denoted by Σ_0. The conservation law can be written

$$P^\mu \equiv \frac{1}{c}\int_{\Sigma_0} T^{\mu\nu} d\Sigma_\nu \;=\; \frac{1}{c}\int T^{\mu 0} dV \quad . \tag{13}$$

Since $T^{\mu\nu}$ has dimensions of energy density, clearly P^μ has the dimensions of momentum, *given our choice of the proportionality constant*. Therefore, P^μ as defined in Eq. (10) represents the **four-momentum of the entire system**, *matter plus fields*, and is a conserved quantity.

• This derivation presumes that there are no contributions to any of the hypsersurface integrations from infinity, which amounts to zero charge and zero field at infinity.

• Drawing the connection to 4-current conservation, we have replaced the 4-current j^μ by the 4-tensor $T^{\mu\nu}$, and have replaced the conserved total (scalar) charge Q by the conserved total four-momentum (vector) P^μ.

The definition of the energy-momentum tensor is not unique. This should be immediately obvious since one can perform gauge transformations of fields and these alter the value of L. Yet, we know from 4-current conservation that such a transformation will not impact the form of the conservation law.

This then provides the clue that a **generalized gauge transformation** might be permissible, provided that it leaves the conservation law in identical form. Consider then

$$T^{\mu\nu} \;\rightarrow\; T^{\mu\nu} + \frac{\partial \psi^{\mu\nu}_{\lambda}}{\partial x_\lambda} \quad . \tag{14}$$

This does not, in general, give a zero four-gradient. However, *if the gauge tensor is anti-symmetric in its last two indices,* ν and λ, i.e., $\psi^{\mu\nu}_{\lambda} = -\psi^{\mu\lambda}_{\nu}$, then the conservation law receives an additional contribution

$$\frac{\partial}{\partial x^\nu}\frac{\partial \psi^{\mu\nu}_{\lambda}}{\partial x_\lambda} \;=\; -\frac{\partial}{\partial x_\nu}\frac{\partial \psi^{\mu\lambda}_{\nu}}{\partial x^\lambda} \;=\; -\frac{\partial}{\partial x^\nu}\frac{\partial \psi^{\mu\nu}_{\lambda}}{\partial x_\lambda} \quad . \tag{15}$$

The last step is just an index re-labelling $\nu \leftrightarrow \lambda$, and thus the derivatives must yield zero due to the antisymmetry property. Thus we again get

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} \;=\; 0 \quad . \tag{16}$$

The four-momentum then maps

$$P^\mu \;\rightarrow\; P^\mu + \frac{1}{c}\int_\Sigma \frac{\partial \psi^{\mu\nu}_{\lambda}}{\partial x_\lambda}\, d\Sigma_\nu \tag{17}$$

The additional term can be expressed using the anti-symmetry and then both of them can be converted into d^4x integrations — they cancel exactly because there are no charges or fields at infinity. Accordingly, the *four-momentum a physical quantity, is uniquely specified.*

1.2 Angular Momentum and $T^{\mu\nu}$

The cross product of space and momentum gives the vector angular momentum, and this can be extended to generate a **4-tensor for angular momentum** (the system total):

$$J^{\mu\nu} \equiv x^\mu P^\nu - x^\nu P^\mu = \frac{1}{c}\int \Big(x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda} \Big) d\Sigma_\lambda \quad . \tag{18}$$

This is inherently anti-symmetric, as should be the case for an extension of the axial vector angular momentum. The **conservation of angular momentum** is naturally expressed via a now familiar 4-gradient identity

$$J^{\mu\nu} = \text{const.} \quad \Rightarrow \quad \frac{\partial}{\partial x^\lambda}\Big(x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda} \Big) = 0 \quad . \tag{19}$$

The second form follows from closing a hypersurface and using Gauss' theorem to convert to a volume integration. This presumes that there is no angular momentum at infinity. Then, using $\partial x^\mu / \partial x^\lambda = \delta^\mu_\lambda$ and the conservation of energy-momentum, $\partial T^{\mu\lambda}/\partial x^\lambda = 0$, this yields

$$0 = \delta^\mu_\lambda T^{\nu\lambda} - \delta^\nu_\lambda T^{\mu\lambda} = T^{\nu\mu} - T^{\mu\nu} \quad \Rightarrow \quad T^{\nu\mu} = T^{\mu\nu} \quad . \tag{20}$$

Thus a fundamental property emerges: the conservation of angular momentum mandates that the **energy-momentum tensor $T^{\mu\nu}$ is symmetric**, i.e., **Hermitian**. The tensor can then be represented by a matrix form

$$T^{\mu\nu} = \begin{pmatrix} U & P_x c/V & P_y c/V & P_z c/V \\ P_x c/V & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{zx} \\ P_y c/V & -\sigma_{xy} & -\sigma_{yy} & -\sigma_{yz} \\ P_z c/V & -\sigma_{zx} & -\sigma_{yz} & -\sigma_{zz} \end{pmatrix} \quad . \tag{21}$$

Here U represents energy density, the $P_i c/V$ are components of momentum density fluxes in different directions, and the σ_{ij} are **momentum shear terms** that define how momentum components in one direction are transported in other directions.

- The couplings of (**x**, **p**) and (t,E) in the covariant $T^{\mu\nu}$ tensor are not accidental, but in part by design, and will facilitate the compact description of angular momentum **L** for light and the general electromagnetic field.