The choice of g is otherwise arbitrary, and falls to unit conventions. In **Gaussian units**, $g = 1/(16\pi)$, and this we take to be a definition. Thus,

$$S_{\rm f} \;=\; -\frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}\,d^4x \;\equiv\; \int L_{\rm f}\,dt \quad , \tag{9}$$

so that

$$L_{\rm f} \;=\; \int \frac{E^2 - B^2}{8\pi}\,dV \tag{10}$$

identifies the Lagrangian $L_{\rm f}$ for the field, is of the dimension of energy.

The total action for charges plus fields in covariant form is then

$$S \;=\; -\int mc\,ds - \int \frac{q}{c}\,A_\mu\,dx^\mu - \frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}\,d^4x \quad . \tag{11}$$

This completes the description of the electromagnetic interaction, and from it one will derive the remaining Maxwell's equations and all physical manifestations of classical electromagnetism.

• Observe that since x^μ scales as the radius r of a volume, the potentials A^μ must decline at least as fast as $1/r$ in large regions in order for the matter-field contribution $S_{\rm mf}$ to remain finite. The fields then drop off as least as rapidly as $1/r^2$ on large scales, so that the pure field contribution $S_{\rm f}$ remains finite.

• Note that for a closed system, A_μ and $F_{\mu\nu}$ constitute the total field, that from internal charges, and that from outside, since the electromagnetic interaction is of infinite range.

3 Four-Current

To set the scene for a covariant form for the remaining two Maxwell's equations that specify how fields are generated by moving charges, we need the formalism of a four-current. This stems from the **charge density** ρ, which for an ensemble of charges q_i at positions $\mathbf{r}_i$ is **L&L Sec. 28**

$$\rho \;=\; \sum_i q_i\,\delta^3(\mathbf{r}-\mathbf{r}_i) \quad\Rightarrow\quad q \;\equiv\; \sum_i q_i \;=\; \sum_i \int q_i\,\delta^3(\mathbf{r}-\mathbf{r}_i)\,dV \quad . \tag{12}$$

The charges themselves are Lorentz invariants, but the charge density is not: only the differential charge element $dq = \rho dV$ is an invariant. Note that the *delta function is of dimensions of an inverse volume.* Thus

$$dq\,dx^\mu \;=\; \rho\,dV\,dx^\mu \;=\; \rho\,\frac{dx^\mu}{dt}\,dV\,dt \tag{13}$$

is a 4-vector, and this leads naturally to the definition of the **four-current**:

$$j^\mu \;=\; \rho u^\mu \;=\; (c\rho,\,\mathbf{j}) \quad \text{for} \quad \mathbf{j} \;=\; \rho\,\mathbf{v} \tag{14}$$

as the conventional 3D **current density**. For our ensemble of moving charges, the three-current density is thus

$$\mathbf{j} \;=\; \sum_i \rho_i\,\mathbf{v}_i\,\delta^3(\mathbf{r}-\mathbf{r}_i) \quad . \tag{15}$$

Clearly the four-current is a *bona fide* four-vector since $dV dt = d^4x/c$ is an invariant, and $d\rho\,dx^\mu$ is a four-vector.

- With this definition, the matter-field term of the action can be recast in truly covariant form. Replacing q by an integral over $dq = \rho dV$, it is

$$S_{\rm mf} \;=\; -\frac{1}{c}\int \rho A_\mu dx^\mu dV \;=\; -\frac{1}{c}\int \rho u^\mu\,A_\mu dV\,dt \;=\; -\frac{1}{c^2}\int j^\mu A_\mu\,d^4x \quad . \tag{16}$$

This is clearly a Lorentz invariant. The total matter/electromagnetic interaction can then be re-written

$$S \;=\; -\sum_i \int m_i c\,ds_i - \frac{1}{c^2}\int j^\mu A_\mu\,d^4x - \frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}\,d^4x \quad . \tag{17}$$

Only the pure matter (first) term explicitly indicates the individual charge label i in a summation, with an equivalent summation subsumed in the four-current. This suffices to describe how fields are generated by charges.

3.1 Equation of Charge Continuity

A fundamental property of electromagnetism is that *the total charge is conserved*. This is akin to mass/energy conservation and in fact the continuity equation to be derived here has a parallel in fluid mechanics. **L&L Sec. 29**

Consider a charge density ρ in a volume enclosed by a surface S. If charge is lost from the volume, it must pass through this surface rather than evaporate. The rate at which it passes through area element $d\mathbf{\Sigma}$ is $-\rho\mathbf{v}\cdot d\mathbf{\Sigma}$. The minus sign is because of loss of charge from the volume. Then

$$\frac{\partial}{\partial t}\int \rho\, dV \;=\; -\oint \rho\mathbf{v}\cdot d\mathbf{\Sigma} \;\equiv\; -\oint \mathbf{j}\cdot d\mathbf{\Sigma} \tag{18}$$

expresses the conservation of charge, where the complete surface is integrated over. The three-current surface integral can be converted back to a volume integral using *Gauss' theorem*:

$$\oint \mathbf{j}\cdot d\mathbf{\Sigma} \;=\; \int \nabla\cdot\mathbf{j}\, dV \quad\Rightarrow\quad \int\left(\nabla\cdot\mathbf{j}+\frac{\partial\rho}{\partial t}\right) dV \;=\; 0 \quad . \tag{19}$$

This is the **equation of charge continuity** in integral form, and since it applies for any volume, the argument of the integral must be zero at all points, implying

$$\frac{\partial j^\mu}{\partial x^\mu} \;\equiv\; \nabla\cdot\mathbf{j}+\frac{\partial\rho}{\partial t} \;=\; 0 \quad , \tag{20}$$

which is the equivalent differential form.

- The same differential form can be inferred from the elemental description of the charge $\rho(\mathbf{r},\, t)$ and current $\mathbf{j}(\mathbf{r},\, t)$:

$$\rho \;=\; \sum_{i=1}^{n} q_i\, \delta^3(\mathbf{r}-\mathbf{r}_i) \quad , \quad \mathbf{j} \;=\; \sum_{i=1}^{n} \rho_i\, \mathbf{v}_i\, \delta^3(\mathbf{r}-\mathbf{r}_i) \quad . \tag{21}$$

During the motion of all the charges (the coordinates of each charge change), the point of observation $\mathbf{r}$ remains fixed. The velocity $\mathbf{v}_i$ of each charge and the average velocity $\mathbf{v}\equiv\langle\mathbf{v}_i\rangle$ of the ensemble can be written

$$\mathbf{v}_i \;=\; \frac{\partial\mathbf{r}_i}{\partial t} \quad , \quad \mathbf{v} \;=\; \frac{1}{n}\sum_{i=1}^{n}\frac{\partial\mathbf{r}_i}{\partial t} \quad . \tag{22}$$

The rate of change of charge density is then

$$\frac{\partial \rho}{\partial t} \;=\; \frac{1}{n}\sum_{i=1}^{n}\frac{\partial \rho}{\partial \mathbf{r}_i}\cdot\frac{\partial \mathbf{r}_i}{\partial t} \;=\; -\frac{\partial \rho}{\partial \mathbf{r}}\cdot\frac{1}{n}\sum_{i=1}^{n}\frac{\partial \mathbf{r}_i}{\partial t} \quad , \tag{23}$$

where the $1/n$ factor introduces an average over all charges. If all the charges move in a bunch together, then we can represent the sum in the last equality by $n\mathbf{v}$, which does not depend on $\mathbf{r}$ and therefore has zero gradient, i.e. divergence. It follows that

$$\frac{\partial \rho}{\partial t} \;=\; -\mathbf{v}\cdot\nabla\rho \;=\; -\nabla(\rho\mathbf{v}) \quad , \tag{24}$$

and we arrive at the differential form of the equation of charge continuity.

- Another finer point concerns the connection between charge conservation and gauge invariance. Consider again the matter-field interaction term in the action S, the only portion of it where such a connection could be evinced. Perform a gauge transformation on the four-potential, $A_\mu \rightarrow A_\mu + \partial f/\partial x^\mu$. The action then changes as follows:

$$S_{\mathrm{mf}} \;=\; -\frac{1}{c^2}\int j^\mu A_\mu \, d^4x \;\rightarrow\; -\frac{1}{c^2}\int j^\mu \left(A_\mu + \frac{\partial f}{\partial x^\mu}\right) d^4x \quad . \tag{25}$$

Now add a term $f\,\partial j^\mu/\partial x^\mu$ to the integrand to form a perfect derivative. This addition is identically zero according to the covariant form of the charge continuity equation. Thus

$$S_{\mathrm{mf}} \;\rightarrow\; -\frac{1}{c^2}\int j^\mu A_\mu \, d^4x - \frac{1}{c^2}\int \frac{\partial(f\,j^\mu)}{\partial x^\mu}\, d^4x \quad . \tag{26}$$

When the variation principle is applied, the second term leads to contributions that are fixed at the endpoints of the spacetime interval segment. Therefore, the gauge-dependent part contributes nothing to the variation of the action, and hence is irrelevant to the actual motion.

Accordingly, the *equation of motion is independent of the choice of gauge*, and *this property is a direct consequence of charge conservation.*

4 Maxwell's Inhomogeneous Field Equations

With an added piece to the action, we can now derive additional equations of motion, and it is sufficient and expedient to examine just the combination of the matter-field $S_{\rm mf}$ and field $S_{\rm f}$ terms. Since we will derive a four-vector variation, the result will be four independent equations of motion, captured in a covariant four-vector equation. The action variation is

L&L Sec. 30

$$\delta S \;=\; -\frac{1}{c}\int\left\{\frac{j^{\mu}}{c}\,\delta A_{\mu}+\frac{1}{8\pi}\,F^{\mu\nu}\delta F_{\mu\nu}\right\}d^4x \;=\; 0 \quad . \tag{27}$$

Remember that we have two variations contributing $\delta F_{\mu\nu}$ that are identical and so a factor of two has been introduced in the second term.

- To be precise, we have already explored charge responses to fields, so now, in order to investigate field responses to charges, we keep the information on charges fixed: this means that no variations in j^{μ} will be admitted.

In the second term, we introduce the derivatives of the four potential, perform an index re-labelling and invoke the anti-symmetry of $F^{\mu\nu}$ to generate the sequence of manipulations

$$F^{\mu\nu}\delta F_{\mu\nu} \;\;\to\;\; \underbrace{F^{\mu\nu}\,\frac{\partial}{\partial x^{\mu}}(\delta A_{\nu})}_{\mu\leftrightarrow\nu}-F^{\mu\nu}\,\frac{\partial}{\partial x^{\nu}}(\delta A_{\mu}) \;=\; -2\,F^{\mu\nu}\,\frac{\partial}{\partial x^{\nu}}(\delta A_{\mu}) \tag{28}$$

This results in

$$\delta S \;=\; -\frac{1}{c}\int\left\{\frac{j^{\mu}}{c}\,\delta A_{\mu}-\frac{1}{4\pi}\,F^{\mu\nu}\,\frac{\partial}{\partial x^{\nu}}(\delta A_{\mu})\right\}d^4x \;=\; 0 \quad . \tag{29}$$

To extract a common increment δA_{μ} under the integral sign, one integrates the second term by parts. Then

$$\delta S \;=\; -\frac{1}{c}\int\left\{\frac{j^{\mu}}{c}+\frac{1}{4\pi}\,\frac{\partial F^{\mu\nu}}{\partial x^{\nu}}\right\}\delta A_{\mu}\,d^4x-\frac{1}{4\pi c}\cancel{\left[\int F^{\mu\nu}\delta A_{\mu}d\Sigma_{\nu}\right]^{\infty}}^{\,0} \quad . \tag{30}$$

This is essentially a 4D version of Gauss' theorem, generating a **hypersurface** integration. We must take the limits of residual term to infinity, since *the electromagnetic field is of infinite range.* This term is then zero, because the fields are necessarily zero at infinity for the total energy to remain finite.

Therefore, the principal of least action yields zero for the factor in the integrand, and a covariant form for the equations of motion for the field emerges:

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} \;=\; -\frac{4\pi}{c}\, j^{\mu} \quad . \tag{31}$$

These inhomogeneous equations describe how *charges produce fields*. The $\mu = 0$ case yields a scalar result involving on the electric field:

$$\nabla\cdot\mathbf{E} \;=\; 4\pi\rho \qquad [\textbf{Maxwell 3}] \quad . \tag{32}$$

Comparing with the magnetic equivalent, this automatically implies the existence if **electric monopoles**, and ingredient that was injected at the outset in the specification of the $S_{\rm mf}$ contribution to the action. The other three equations form a cyclic ensemble, with algebra like:

$$\begin{aligned} \cancel{\frac{\partial F^{11}}{\partial x}} + \frac{\partial F^{12}}{\partial y} + \frac{\partial F^{13}}{\partial z} + \frac{1}{c}\frac{\partial F^{10}}{\partial t} \;&=\; -\frac{4\pi}{c}\, j^1 \\ \Rightarrow \quad -\frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} + \frac{1}{c}\frac{\partial E_x}{\partial t} \;&=\; -\frac{4\pi}{c}\, j_x \quad . \end{aligned} \tag{33}$$

Combining them results in a vector equation known as **Ampere's Law**:

$$\nabla\times\mathbf{B} \;=\; \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\,\mathbf{j} \qquad [\textbf{Maxwell 4}] \quad . \tag{34}$$

This completes **Maxwell's Equations**, the governing field equations for electromagnetism. From these four differential equations, all observable properties of this classical theory can be derived.

• Observing the presence of time derivatives of each field in a conjugate sense in the two curl equations, but with the opposite sign, we note that this structure provides the essential mathematical character to generate electromagnetic waves as a solution in free space, removed from the locales of charges that generate E/M fields, and respond to them.

The integral forms for these two equations are quickly derived. Applying Gauss' theorem to the monopole equation, we have

$$\oint \mathbf{E}\cdot d\mathbf{\Sigma} \;=\; \int \nabla\cdot\mathbf{E}\, dV \;=\; 4\pi \int \rho\, dV \;=\; 4\pi q \quad . \tag{35}$$

Accordingly, the integrated **electric flux** over a closed surface equals 4π times the charge in the enclosed volume. From this, the Coulomb potential naturally emerges, for a point charge at the center of a spherical surface.

Ampere's Law clearly needs to be manipulated using Stoke's theorem, and the result defines the **magnetic circulation** around a loop:

$$\oint \mathbf{B} \cdot d\mathbf{l} \;=\; \frac{4\pi}{c} \int \left(\mathbf{j} + \frac{1}{4\pi}\,\frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{\Sigma} \quad . \tag{36}$$

This equals the flux of current $\mathbf{j}$ and **displacement current** ($\propto \partial\mathbf{E}/\partial t$) integrated over the contacting surface.

- On a final note, we observe that the covariant form of the inhomogeneous Maxwell's equations can be differentiated to obtain the continuity equation describing charge conservation:

$$\frac{4\pi}{c}\,\frac{\partial j^\mu}{\partial x^\mu} \;=\; -\frac{\partial^2 F^{\mu\nu}}{\partial x^\nu \partial x^\mu} \;=\; \frac{\partial^2 F^{\nu\mu}}{\partial x^\nu \partial x^\mu} \;=\; 0 \quad . \tag{37}$$

4.1 Energy Density and Energy Flux

The time variations inherent in two of Maxwell's equations can be combined in the following informative way. Form **L&L Sec. 31**

$$\mathbf{B} \cdot \underbrace{\left\{ \nabla \times \mathbf{E} + \frac{1}{c}\,\frac{\partial \mathbf{B}}{\partial t} \right\}}_{\text{Faraday's Law}} - \mathbf{E} \cdot \underbrace{\left\{ \nabla \times \mathbf{B} - \frac{1}{c}\,\frac{\partial \mathbf{E}}{\partial t} - \frac{4\pi}{c}\,\mathbf{j} \right\}}_{\text{Ampere's Law}} \;\equiv\; 0 \quad . \tag{38}$$

This rearranges to

$$\frac{1}{c}\,\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{c}\,\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \;=\; -\frac{4\pi}{c}\,\mathbf{j} \cdot \mathbf{E} - \Big\{ \mathbf{B} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{B} \Big\} \quad . \tag{39}$$

The factor inside the curly braces is just the divergence $\nabla \cdot (\mathbf{E} \times \mathbf{B})$, so we then have

$$\frac{1}{2c}\,\frac{\partial}{\partial t}\Big(E^2 + B^2 \Big) \;=\; -\frac{4\pi}{c}\,\mathbf{j} \cdot \mathbf{E} - \nabla \cdot (\mathbf{E} \times \mathbf{B}) \quad . \tag{40}$$

To render this in final form, multiply through by $c/4\pi$, yielding

$$\frac{\partial}{\partial t}\left(\frac{E^2 + B^2}{8\pi} \right) \;=\; -\mathbf{j} \cdot \mathbf{E} - \nabla \cdot \mathbf{S} \quad \text{for} \quad \mathbf{S} \;=\; \frac{c}{4\pi}\,(\mathbf{E} \times \mathbf{B}) \quad . \tag{41}$$

The quantity $\mathbf{S}$ is called the **Poynting vector**.

To interpret this equation, we integrate over a volume V and use Gauss' theorem to convert the divergence term to a surface integral:

$$\frac{\partial}{\partial t}\int\Big(\frac{E^2+B^2}{8\pi}\Big)\,dV \;=\; -\int \mathbf{j}\cdot\mathbf{E}\,dV - \oint \mathbf{S}\cdot d\mathbf{\Sigma} \quad . \tag{42}$$

Here $d\mathbf{\Sigma}$ is a surface element. Now the term with the current density in it is just a sum of $q\mathbf{v}\cdot\mathbf{E}$ over all the charges. This is the total rate of work done on the charge ensemble, and therefore the rate of change of kinetic energy. If the volume is infinite, then the Poynting vector term is identically zero as the fields are zero there. Then,

$$\frac{d}{dt}\left\{\int^{\infty}\frac{E^2+B^2}{8\pi}\,dV + \sum \mathcal{E}_{\rm kin}\right\} \;=\; 0 \quad . \tag{43}$$

This is obviously an expression of <u>energy conservation</u>. Therefore

$$U_{\rm em} \;=\; \frac{E^2+B^2}{8\pi} \tag{44}$$

is interpreted as the **energy density** of the electromagnetic field, its energy per unit volume. If we consider finite volumes, then Eq. (43) becomes

$$\frac{d}{dt}\left\{\int\frac{E^2+B^2}{8\pi}\,dV + \sum_V \mathcal{E}_{\rm kin}\right\} \;=\; -\oint \mathbf{S}\cdot d\mathbf{\Sigma} \quad . \tag{45}$$

The kinetic energy contribution comes only from particles in the volume. Thus this new form of energy conservation captures both the field and particle content, and what is lost within the volume must pass through the enclosing surface. The Poynting vector thus represents a **energy flux density**, i.e. the amount of energy passing per unit area per unit time through the surface.

- If the loss is conveyed at speed c, then the flux density is an energy density per speed c, and so is a **momentum density**.

• It is instructive to demonstrate that the fields have the correct units to accommodate $(E^2 + B^2)/8\pi$ being an energy density. This is quite simple. From the electric potential, the field has dimensions

$$\dim(E) \;=\; \frac{\text{energy}}{\text{length} \cdot \text{charge}} \quad . \tag{46}$$

From the gyroradius $r_g = p_\perp c/qB$ for helical motion in a uniform magnetic field, we discern that the magnetic field has dimensions

$$\dim(B) \;=\; \frac{\text{energy}}{\text{length} \cdot \text{charge}} \quad , \tag{47}$$

i.e. <u>the same as the electric field</u> (highlighting a benefit of Gaussian units). These can then be squared and combined, producing a square of the charge in the denominator. Then, remembering that the Coulomb potential energy is $\sim Q^2/r$, we determine that the square of the charge has dimensions of energy times length. It follows that

$$\dim(E^2 + B^2) \;=\; \left(\frac{\text{energy}}{\text{length} \cdot \text{charge}} \right)^2 \;=\; \frac{\text{energy}}{(\text{length})^3} \quad . \tag{48}$$

Clearly this is an energy density.