2.1 Field Invariants

We can identify key Lorentz invariants by forming scalars from the electromagnetic tensor $F_{\mu\nu}$. There are two such invariants that are linearly independent and that are quickly obtained: **L&L Sec. 25**

$$F_{\mu\nu} F^{\mu\nu} \; = \; 2\Bigl(\mathbf{B}^2 - \mathbf{E}^2\Bigr) \quad , \quad \det F \; = \; \Bigl(\mathbf{E}\cdot\mathbf{B}\Bigr)^2 \quad . \tag{24}$$

The first, the square of F, is an invariant because the contraction process, when applied to Lorentz transformation tensors Λ_α^β, suitably yields unit tensors. The second results because the determinant of the Λ_α^β tensors is unity, and determinants multiply.

- Both these invariants can be established directly using Eq. (20).

From the matrix forms for the electromagnetic field tensor,

$$F_{\mu\nu} F^{\mu\nu} \; = \; \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \langle\cdot\rangle \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \; , \tag{25}$$

and it is then easily seen that this is just $2(\mathbf{B}^2 - \mathbf{E}^2)$. The invariance of this quantity just expresses conservation of "internal" energy: while boosts can enhance the total energy perceived by an observer, the relative content in the magnetic and electric components is fixed.

- The invariance of $\mathbf{E}\cdot\mathbf{B}$ connects to the vector identity

$$(\mathbf{E}\cdot\mathbf{B})^2 \; = \; E^2\, B^2 - (\mathbf{E}\times\mathbf{B})\cdot(\mathbf{E}\times\mathbf{B}) \quad , \tag{26}$$

and essentially indicates that the zero rest mass of the electromagnetic field is conserved under Lorentz transformations: the first term on the RHS represents energy content, and the second, Poynting flux or momentum content.

$\ast$ Note that the energy "character" of the electromagnetic field will be established in due course, requiring the identification of how charges generate fields: e.g. learning that q^2/r has the dimensions of energy.

• Interpreting these invariants is enhanced using *gedanken experiments* involving static charge distributions with zero current, or counterstreaming currrents of charges of opposite sign but zero net charge. Both circumstances possess $\mathbf{E}\cdot\mathbf{B} = 0$, which cannot be transformed away by a boost.

∗ The zero values of both invariants constitutes the statement that light is perceived as light in all frames of reference.

An elegant path to identifying the two electromagnetic invariants, and the fact that there are only two, is offered in L&L. Consider the complex vector $\mathbf{F} = \mathbf{E} + i\mathbf{B}$. The Lorentz transformation properties of the fields naturally can be expressed, not surprisingly, via **rapidities** χ with $\tanh\chi = \beta$ or $\cosh\chi = \gamma$. Thus

$$\begin{aligned} E'_x = E_x \quad , \quad E'_y \; &= \; \gamma(E_y - \beta B_z) \quad , \quad E'_z \; = \; \gamma(E_z + \beta B_y) \quad , \\ B'_x = B_x \quad , \quad B'_y \; &= \; \gamma(B_y + \beta E_z) \quad , \quad B'_z \; = \; \gamma(B_z - \beta E_y) \quad , \end{aligned} \tag{27}$$

for a boost in the x-direction can be cast in the form

$$\begin{aligned} F_x &= F'_x \quad , \\ F_y &= F'_y \cosh\chi - iF'_z \sinh\chi \;=\; F'_y \cos i\chi - F'_z \sin i\chi \quad , \\ F_z &= F'_z \cosh\chi + iF'_y \sinh\chi \;=\; F'_z \cos i\chi + F'_y \sin i\chi \quad . \end{aligned} \tag{28}$$

This is just a rotation through imaginary angles in the (x, t)-plane. Such an inference can be extended to the other space dimensions, and include space rotations so that the Lorentz transformation is generally expressed as a sequence of rotations in 3D space. The only invariant of a vector $\mathbf{F}$ under such rotations is its magnitude. Accordingly

$$\mathbf{F}\cdot\mathbf{F} \;=\; \Bigl(E^2 - B^2\Bigr) + 2i\,\mathbf{E}\cdot\mathbf{B} \tag{29}$$

must be *the only invariant of the electromagnetic field*. Taking real and imaginary results yields the **complete set** of two E/M invariants given in Eq. (24).

6. THE ELECTROMAGNETIC FIELD EQUATIONS

Matthew Baring — Lecture Notes for PHYS 532, Spring 2023

1 Maxwell's Pair of Homogeneous Equations

The first pair of field equations known as **Maxwell's Equations** can be constructed using the information of the matter-field interaction. This led to the identification of measurable fields **E** and **B** that can be expressed in terms of the four-potential in the Lagrangian: **L&L Sec. 26**

$$\mathbf{E} \;=\; -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi \quad , \quad \mathbf{B} \;=\; \nabla \times \mathbf{A} \quad . \tag{1}$$

If we take the curl of the electric field, since the curl of a gradient is zero, then we have **Faraday's Law** of induction:

$$\nabla \times \mathbf{E} \;=\; -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad [\textbf{Maxwell 1}] \quad . \tag{2}$$

Eventually, this will reveal that time-varying currents can generate time-varying **E** fields, and we have the basis for transmission of electromagnetic waves. Taking the divergence of the magnetic field also simply yields zero:

$$\nabla \cdot \mathbf{B} \;=\; 0 \qquad [\textbf{Maxwell 2}] \quad . \tag{3}$$

This differs in form from the divergence of **E**, which for a Coulomb field we know is non-zero. Accordingly, this equation expresses the **non-existence of magnetic monopoles**. To devise a theory that predicts magnetic monopoles, the matter-field action must be modified from our adopted $S_{\rm mf}$ form.

- These two homogeneous equations do not describe how charges generate fields. They also do not determine $\partial\mathbf{E}/\partial t$. Thus, we know we need more!

These two equations, one scalar, one vector, are linear derivatives of the fields, and so are second order derivatives of the four-potential. A covariant form for these Maxwell equations follows from the E/M field tensor:

$$F_{\alpha\beta} \;=\; \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} \quad\Rightarrow\quad \frac{\partial F_{\alpha\mu}}{\partial x^\nu} + \frac{\partial F_{\mu\nu}}{\partial x^\alpha} + \frac{\partial F_{\nu\alpha}}{\partial x^\mu} \;=\; 0 \quad . \tag{4}$$

This is a cyclic divergence that is anti-symmetric in all three indices. The only components that are not zero are when $\alpha \neq \mu \neq \nu$. Insertion of the field tensor reveals that there are only 4 independent equations, matching the number in Maxwell 1+2. Note that one could take covariant derivatives of the Lorentz force equation

$$mc^2\,\frac{du_\mu}{ds} \;=\; q\,F_{\mu\nu}u^\nu \tag{5}$$

to establish the same cyclic identity using four-accelerations.

The alternative to the differential form of these two Maxwell equations is given by integral forms, obtained by methods of vector calculus. The magnetic field can be integrated over a volume or equivalently the surface integral encapsulating that volume:

$$0 \;=\; \int \nabla\cdot\mathbf{B}\,dV \;=\; \oint \mathbf{B}\cdot d\mathbf{\Sigma} \quad , \tag{6}$$

where $d\mathbf{\Sigma}$ is the vector surface element for the volume. This is **Gauss' Law** and states that the net magnetic flux through a closed surface is zero, a consequence of the non-existence of magnetic monopoles.

The electric field is manipulated by **Stoke's theorem** when morphing a surface integral into a line integral:

$$-\frac{1}{c}\frac{\partial}{\partial t}\int \mathbf{B}\cdot d\mathbf{\Sigma} \;=\; \int \nabla\times\mathbf{E}\cdot d\mathbf{\Sigma} \;=\; \oint \mathbf{E}\cdot d\mathbf{l} \quad . \tag{7}$$

This applies to non-closed surfaces and so the LHS is not necessarily zero. Yet the line integral on the RHS, which is called the **electric circulation**, does in fact go to zero (trivially) when the surface is closed.

2 The Action of the Electromagnetic Field

To derive the other two Maxwell's equations we need more information, and this must concern the electromagnetic field itself. Since light transmits energy and momentum from one place to another, an experimental fact, the E/M field on its own must contribute to the Lagrangian, Hamiltonian and action $S = S_{\rm m} + S_{\rm mf} + S_{\rm f}$. Without such a contribution, we cannot derive the existence of electromagnetic waves. **L&L Sec. 27**

The contribution $S_{\rm f}$ of the field to the action therefore cannot depend on the charge. Thus, couplings of the form $[qA^\mu]^n$ are excluded. The following elements guide the natural choice for this portion of the action:

- The **E** and **B** fields satisfy the **superposition principle**, namely that the field produced by a system of charges in motion is equal to the sum of the fields produced by each individual charge.
- This linearity imposes the constraint of linearity on the field equations, including the equation of motion for charges. [already established]
- The information on the field $F^{\mu\nu}$ must then be inherently quadratic, so that Hamilton's principle will then derive linear appearance of $F^{\mu\nu}$. If it is of higher order, then linearity is not possible. If it is of linear order, then charges will not generate fields in a linear manner.
- In a quadratic construction, the potentials cannot appear since they are not uniquely-determined. This is not an issue for the linear S_{mf} term because gauges are eliminate via integrations by parts.

This shopping list suggests a choice involving the contraction $F_{\mu\nu}F^{\mu\nu}$. We already know that this possesses the dimensions of energy density. This provides an additional physics motivation for the quadratic form: that it naturally leads to the formation of energy in the Lagrangian. Thus we posit:

$$S_{\rm f} \; = \; -\frac{g}{c} \int F_{\mu\nu}F^{\mu\nu} \, d^4x \quad , \tag{8}$$

where g is a constant, inferred to be dimensionless because the integrand $F_{\mu\nu}F^{\mu\nu} = 2(B^2 - E^2)$ has units of energy density. Since the electric field can be made arbitrarily large by rapidly varying the vector potential, $\partial\mathbf{A}/\partial t$, the sign of g must be positive in order to guarantee a minimum in the action.

The choice of g is otherwise arbitrary, and falls to unit conventions. In **Gaussian units**, $g = 1/(16\pi)$, and this we take to be a definition. Thus,

$$S_{\rm f} \;=\; -\frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}\, d^4x \;\equiv\; \int L_{\rm f}\, dt \quad , \tag{9}$$

so that

$$L_{\rm f} \;=\; \int \frac{E^2 - B^2}{8\pi}\, dV \tag{10}$$

identifies the Lagrangian $L_{\rm f}$ for the field, is of the dimension of energy.

The total action for charges plus fields in covariant form is then

$$S \;=\; -\int mc\, ds - \int \frac{q}{c}\, A_\mu\, dx^\mu - \frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}\, d^4x \quad . \tag{11}$$

This completes the description of the electromagnetic interaction, and from it one will derive the remaining Maxwell's equations and all physical manifestations of classical electromagnetism.

- Observe that since x^μ scales as the radius r of a volume, the potentials must decline at least as fast as $1/r$ in large regions in order for the matter-field contribution to remain finite. The fields then drop off as least as rapidly as $1/r^2$ on large scales, so that the pure field contribution $S_{\rm f}$ remains finite.

- Note that for a closed system, A_μ and $F_{\mu\nu}$ constitute the total field, that from internal charges, and that from outside, since the electromagnetic interaction is of infinite range.