5. THE ELECTROMAGNETIC FIELD TENSOR

Matthew Baring — Lecture Notes for PHYS 532, Spring 2023

1 The Electromagnetic Field Tensor

We have discovered in Chapter 4 the equation of motion of charges in electromagnetic fields in familiar three-dimensional vector form. Here we seek to extend this to covariant form in spacetime. The coupling constant between the four-force and the four-velocity will lead to the definition of the electromagnetic field tensor, $F^{\mu\nu}$.

Given the two contributions to the action thus far posited, the matter term, and the matter-field interaction, the principle of least action states that **L&L Sec. 23**

$$\delta S \;=\; \delta \int_a^b \left(-mc\, ds - \frac{q}{c}\, A_\mu\, dx^\mu \right) \;=\; 0 \tag{1}$$

The line element can be expressed as $ds = \sqrt{dx_\mu dx^\mu}$. Then the variation can be brought inside the integration thus:

$$\delta S \;=\; -\int_a^b \left(mc\, \frac{dx_\mu d(\delta x^\mu)}{ds} + \frac{q}{c}\, A_\mu\, d(\delta x^\mu) + \frac{q}{c}\, \delta A_\mu\, dx^\mu \right) \;=\; 0 \;\; . \tag{2}$$

Now integrate the first two terms by parts, for both of which the evaluations at the fixed endpoints yield zero. The first term is proportional to $u_\mu d(\delta x^\mu)$, so that it morphs into a term proportional to $du_\mu\, \delta x^\mu$. The second term morphs into one proportional to $\delta x^\mu dA_\mu$. Thus,

$$\int_a^b \left(mc\, \delta x^\mu\, du_\mu + \frac{q}{c}\, \delta x^\mu\, dA_\mu - \frac{q}{c}\, \delta A_\nu\, dx^\nu \right) - \cancel{\left[\left(mcu_\mu + \frac{q}{c} A_\mu \right) \delta x^\mu \right]_a^b} \;=\; 0 \; . \tag{3}$$

Observe that the dummy summation index μ has been changed to ν in the third term, to precipitate a convenience that will shortly become obvious. The vector potential factors in the last two terms are manipulated by the chain rule for differentiation:

$$dA_\mu \;\rightarrow\; \frac{\partial A_\mu}{\partial x^\nu}\,dx^\nu \quad , \quad \delta A_\nu \;\rightarrow\; \frac{\partial A_\nu}{\partial x^\mu}\,\delta x^\mu \tag{4}$$

Therefore,

$$\int_a^b \delta x^\mu \left(mc\,du_\mu + \frac{q}{c}\,\frac{\partial A_\mu}{\partial x^\nu}\,dx^\nu - \frac{q}{c}\,\frac{\partial A_\nu}{\partial x^\mu}\,dx^\nu \right) \;=\; 0 \quad . \tag{5}$$

Now perform the replacements $d\mu_\mu \rightarrow d\mu_\mu/ds \times ds$ and $dx^\nu \rightarrow u^\nu ds$. The integration simplifies to

$$\int_a^b \delta x^\mu \left[mc\,\frac{du_\mu}{ds} - \frac{q}{c}\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) u^\nu \right] ds \;=\; 0 \quad . \tag{6}$$

Again, since this applies to arbitrary δx^μ, the factor in square brackets must be zero, yielding the covariant equation of motion:

$$\boxed{mc\,\frac{du_\mu}{ds} \;=\; \frac{q}{c}\,F_{\mu\nu}\,u^\nu \quad \text{for} \quad F_{\mu\nu} \;=\; \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}} \quad . \tag{7}$$

The anti-symmetric tensor $F_{\mu\nu}$ is called the **electromagnetic field tensor**; its components will be detailed shortly.

- Eq. (7) is the covariant form of the Lorentz force equation, expressing the rate of change of 4-momentum in terms of electromagnetic forces; we have derived it before. It *really only expresses three independent equations*, an assertion that can be proven using the contraction

$$0 \;=\; \frac{mc}{2}\,\frac{d(u^\mu u_\mu)}{ds} \;=\; mc\,u^\mu \frac{du_\mu}{ds} \;=\; \frac{q}{c}\,u^\mu\,F_{\mu\nu}\,u^\nu \;=\; -\frac{q}{c}\,u^\nu\,F_{\nu\mu}\,u^\mu \quad . \tag{8}$$

This uses $u^\mu u_\mu = 1$ on the LHS and the anti-symmetry of the field tensor on the RHS to show that the four equations are linearly dependent.

The anti-symmetry of the field tensor guarantees that $F_{00} = F_{11} = F_{22} = F_{33} = 0$, i.e. it is traceless, and that it possesses only six independent non-zero components. Using $A_\mu = (\phi, -\mathbf{A})$, algebraic evaluation yields

$$\begin{aligned}
F_{01} &= -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} = E_x \\
F_{02} &= -\frac{1}{c}\frac{\partial A_y}{\partial t} - \frac{\partial \phi}{\partial y} = E_y \\
F_{03} &= -\frac{1}{c}\frac{\partial A_z}{\partial t} - \frac{\partial \phi}{\partial z} = E_z \\
F_{12} &= -\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} = -B_z \\
F_{13} &= \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = B_y \\
F_{23} &= -\frac{\partial A_z}{\partial y} + \frac{\partial A_y}{\partial z} = -B_x \quad .
\end{aligned} \tag{9}$$

The matrix representation of the covariant field tensor is then

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \quad . \tag{10}$$

Raising both indices changes the sign of the electric field components but keeps the magnetic field components unaltered. The contravariant form is

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad . \tag{11}$$

In terms of the polar/axial vector designation, we have $F_{\mu\nu} = (\mathbf{E}, \mathbf{B})$ and $F^{\mu\nu} = (-\mathbf{E}, \mathbf{B})$, perhaps an easier way to remember the matrix forms. Thus, $\mathbf{E}$ is a <u>polar vector</u>, while $\mathbf{B}$ is an <u>axial vector</u>.

The force consists of two parts on the RHS, the first being independent of the velocity of the particle, and proportional to the charge. The ratio of the two is *defined to be* the **electric field intensity**:

$$\mathbf{E} \;=\; -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi \quad . \tag{15}$$

The second part of the force is proportional to the velocity, but is always perpendicular to it. Thus we define the **magnetic field intensity**:

$$\mathbf{B} \;=\; \nabla \times \mathbf{A} \quad . \tag{16}$$

A general electromagnetic field is thus a superposition of an electric and a magnetic field. The equation of motion then assumes the form

$$\frac{d\mathbf{p}}{dt} \;=\; q\left\{\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right\} \quad , \tag{17}$$

which is termed the **Newton-Lorentz force equation**.

• If, as before, we impose the path in the action pertaining to the exact equation of motion, but then choose to vary the endpoint b of the path integral, then the perturbation yields

$$\delta S \;=\; -\Bigl(mcu_\mu + \frac{q}{c} A_\mu \Bigr) \delta x^\mu \quad \Rightarrow \quad \frac{\partial S}{\partial x^\mu} \;=\; -\Bigl(p_\mu + \frac{q}{c}\, A_\mu \Bigr) \quad . \qquad (12)$$

This gradient is just the negative of the **canonical momentum** of a particle, and so one has

$$P^\mu \;\equiv\; -\frac{\partial S}{\partial x_\mu} \;=\; p^\mu + \frac{q}{c}\, A^\mu \;=\; \biggl(\frac{\mathcal{E}_{\rm K} + q\phi}{c} ,\, \mathbf{p} + \frac{q}{c}\, \mathbf{A} \biggr) \quad , \qquad (13)$$

and $P^\mu P_\mu$ can be used to express the square of the relativistic Hamiltonian.

2 Lorentz Transformation of the Field

The vector potential is a true four-vector, and so its Lorentz transformation is of familiar form. For boosts parallel to the x-axis, **L&L Sec. 24**

$$\phi \;=\; \gamma \Bigl(\phi' + \beta A'_x \Bigr) \,, \quad A_x \;=\; \gamma \Bigl(A'_x + \beta \phi' \Bigr) \,, \quad A_y \;=\; A'_y \,, \quad A_z \;=\; A'_z \quad . \qquad (14)$$

Since $\mathbf{E}$ and $\mathbf{B}$ are not space components of any four-vector, their transformation is more complicated. It is possible to derive their transformation relations using these vector potential identities. However, as they are components of the electromagnetic field tensor, this affords an expedient path to proceed. Let $\mathbf{v} = \beta c\, \hat{x}$ be the velocity of the K' frame in inertial frame K.

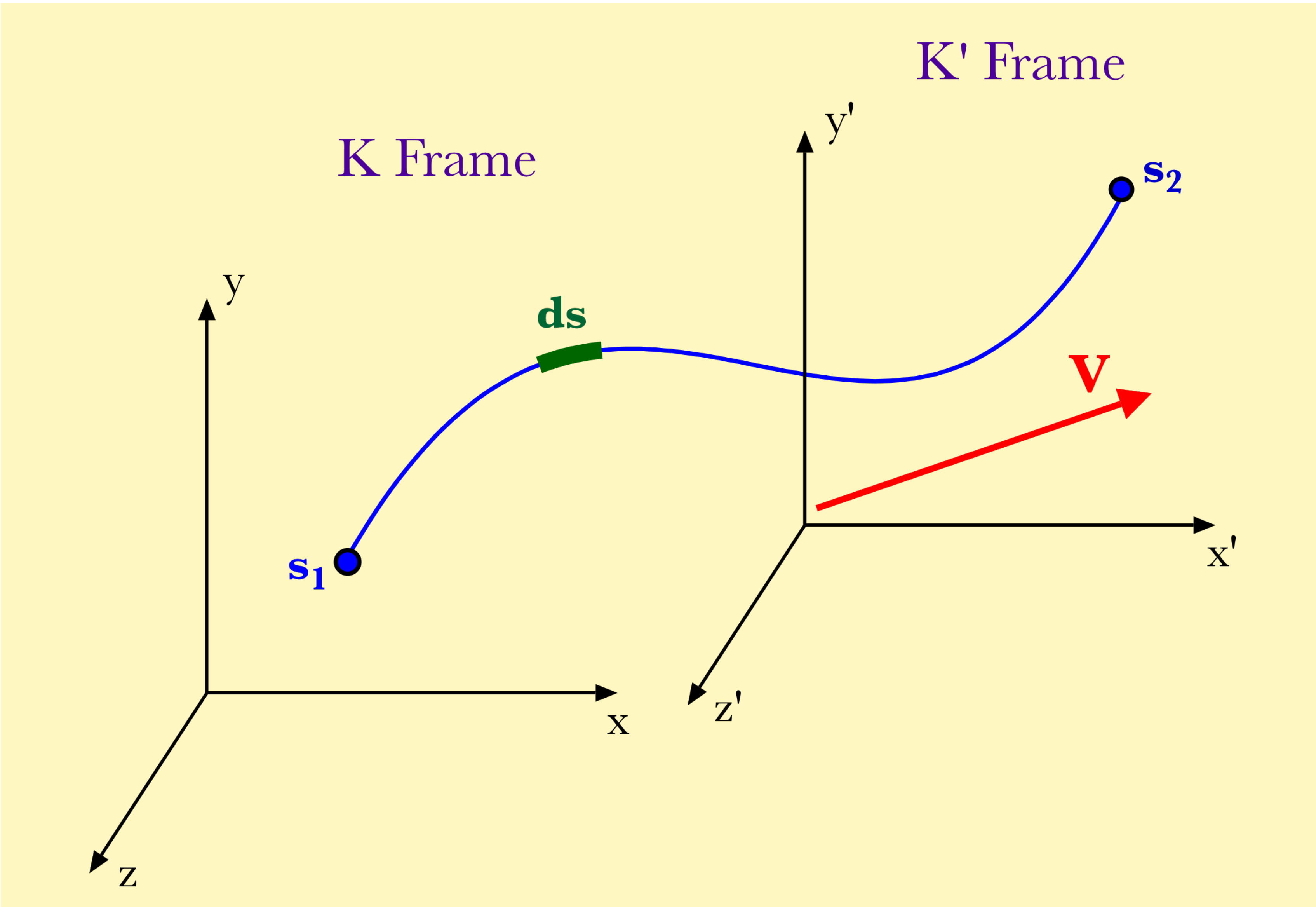
Plot: Draw K and K' frames and relative speed $\mathbf{v}$.

From the Lorentz transformation relation for the electromagnetic tensor,

$$F'^{\mu\nu} \;=\; \frac{\partial x'^\mu}{\partial x^\alpha}\, F^{\alpha\beta}\, \frac{\partial x'^\nu}{\partial x^\beta} \;\equiv\; \mathbf{\Lambda}^\mu_\alpha\, F^{\alpha\beta}\, \mathbf{\Lambda}^\nu_\beta \quad . \qquad (15)$$

Here $\mathbf{\Lambda}^\mu_\alpha$ is the Lorentz transformation matrix for the inverse boost, from the K' frame to the K frame, and so is obtained from our habitual form

Two Inertial Frames in Relative Motion



by setting $\beta \to -\beta$. Noting that $\mathbf{\Lambda}^T = \mathbf{\Lambda}$ for the symmetric (Hermitian) Lorentz transformation matrix, we can build the matrix algebra thus:

$$F^{\alpha\beta}\,\mathbf{\Lambda}_{\beta}^{\nu} \;\to\; \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \cdot \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \tag{16}$$

using the contravariant form $F^{\alpha\beta}$. The next step yields $\overline{\mathbf{\Lambda}}_{\mu}^{\alpha}\,F_{\alpha\beta}\,\overline{\mathbf{\Lambda}}_{\nu}^{\beta}$:

$$\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma\beta E_x & -\gamma E_x & -E_y & -E_z \\ \gamma E_x & -\gamma\beta E_x & -B_z & B_y \\ \gamma(E_y - \beta B_z) & \gamma(B_z - \beta E_y) & 0 & -B_x \\ \gamma(E_z + \beta B_y) & -\gamma(B_y + \beta E_z) & B_x & 0 \end{pmatrix} \quad . \tag{17}$$

This algebra gives an upper half evaluation

$$\begin{pmatrix} \cancel{\gamma^2\beta E_x - \gamma^2\beta E_x} & \gamma^2\beta^2 E_x - \gamma^2 E_x & \gamma(-E_y + \beta B_z) & -\gamma(E_z + \beta B_y) \\ \dots & \cancel{\gamma^2\beta E_x - \gamma^2\beta E_x} & \gamma(-B_z + \beta E_y) & \gamma(B_y + \beta E_z) \\ \dots & \dots & 0 & -B_x \\ \dots & \dots & \dots & 0 \end{pmatrix} , \tag{18}$$

with anti-symmetry constraining the remaining elements. Identifying each of these elements with the fields in the $F'^{\mu\nu}$ <u>contravariant</u> representation, we arrive at the **boost relations for electromagnetic fields**:

$$\begin{aligned} E'_x = E_x \quad , \quad E'_y \;&=\; \gamma(E_y - \beta B_z) \quad , \quad E'_z \;=\; \gamma(E_z + \beta B_y) \quad , \\ B'_x = B_x \quad , \quad B'_y \;&=\; \gamma(B_y + \beta E_z) \quad , \quad B'_z \;=\; \gamma(B_z - \beta E_y) \quad . \end{aligned} \tag{19}$$

A compact representation of these relations for a pure boost with general velocity $\mathbf{v} = c\boldsymbol{\beta}$ for the motion of the K' frame in the K frame, is that the field components transform as

$$\begin{aligned} \mathbf{E}'_{\|} \;&=\; \mathbf{E}_{\|} \quad , \quad \mathbf{E}'_{\perp} \;=\; \gamma\Big(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B}_{\perp}\Big) \quad , \\ \mathbf{B}'_{\|} \;&=\; \mathbf{B}_{\|} \quad , \quad \mathbf{B}'_{\perp} \;=\; \gamma\Big(\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}_{\perp}\Big) \quad . \end{aligned} \tag{20}$$

Yet another form that is fully general, and that does not employ the $\perp, \|$ notation is from Jackson p.558 (3rd Edition):

$$\begin{aligned} \mathbf{E}' &= \gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma+1}\, \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}) \quad , \\ \mathbf{B}' &= \gamma(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma+1}\, \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B}) \quad . \end{aligned} \tag{21}$$

These alternative forms can be used to ascertain whether the either electric or magnetic field (but not both) can be eliminated by a Lorentz transformation.

- Observe the symmetric role played by $\mathbf{E}$ and $-\mathbf{B}$. It is immediately realized that there is *no such thing as a pure electric or magnetic field in all frames*; they are not Lorentz invariants.

- Either form of these relations can be used to show that if $\mathbf{E}$ and $\mathbf{B}$ are parallel, then no boost can eliminate either the electric or the magnetic field entirely. Conversely, they can also be used to determine the special boost velocity $\boldsymbol{\beta}$ that yields $\mathbf{E}'$ and $\mathbf{B}'$ fields parallel to each other.

- For a physical understanding of these transformations, consider a capacitance established by two planes of charges separated by distance d in the K frame, and oriented perpendicular to the x direction. The electric field $E = 4\pi\sigma$ is uniform between these planes whose surface charge density is σ (from Maxwell's equations, to come). From the perspective of the K' frame, the capacitor moves with speed v along x, so that the planes are now separated by d/γ. Yet, the surface charge density is unchanged $\sigma' = \sigma$, since the areas do not change, nor does the total charge on each plane. Then

R & L, Sec. 4.5

$$\mathbf{E}'_{\|} = \mathbf{E}_{\|} \tag{22}$$

authomatically follows.

- Now orient the capacitor planes parallel to the x axis. Since the lengths are contracted, the charge density is increased to $\sigma' = \gamma\sigma$ in the K' frame. We have a **surface current** density $\mu' = -\sigma' v \equiv -\gamma\sigma v$ in the K' frame, giving rise to a magnetic field $B'_z = -(4\pi/c)\mu'$ in the z-direction. Thus,

$$\mathbf{E}'_{\perp} = \gamma \mathbf{E}_{\perp} \quad , \quad \mathbf{B}'_{\perp} = -\gamma \boldsymbol{\beta} \times \mathbf{E}_{\perp} \quad . \tag{23}$$

Other constructions combined with superposition of fields can yield the total ensemble of transformation elements.