3.2 Motion in Uniform Magnetic Fields

The key difference of motion in a magnetic field from that in an electric field is that now the force is orthogonal to the instantaneous motion, and so does no work. Thus a charge's energy $\mathcal{E}$ is conserved. Here we choose the uniform $\mathbf{B}$ field along the z-axis. The Lorentz force equation of motion (EOM) is **L&L Sec. 21**

$$\frac{d}{dt}\left(\frac{\mathcal{E}\mathbf{v}}{c^2}\right) \equiv \frac{d\mathbf{p}}{dt} = \frac{q}{c}\,\mathbf{v}\times\mathbf{B} \quad . \tag{45}$$

Here we have used the identity $\mathbf{p} = \mathcal{E}\mathbf{v}/c^2$. The energy $\mathcal{E} \equiv \mathcal{E}_{\rm K}$ can be passed outside the time derivative, and the EOM can be recast as

$$\frac{d\mathbf{v}}{dt} = \omega\,\mathbf{v}\times\hat{\mathbf{B}} \quad , \quad \omega = \frac{qcB}{\mathcal{E}} \quad . \tag{46}$$

This is simply an EOM for a centripetal acceleration so that ω represents the angular frequency of the motion. Resolving it into pertinent components,

$$\dot{v}_x = \omega\,v_y \quad , \quad \dot{v}_y = -\omega\,v_x \quad , \quad \dot{v}_z = 0 \quad . \tag{47}$$

For the first integration of the EOM, we combine the motions transverse to $\mathbf{B}$ using complex variables to quickly solve the simultaneous ODEs:

$$\frac{d}{dt}\left(v_x + iv_y\right) = -i\omega\left(v_x + iv_y\right) \quad \Rightarrow \quad v_x + iv_y = v_c\,e^{-i(\omega t+\alpha)} \quad . \tag{48}$$

Thus two constants of integration appear that serve to define the initial velocity components in the (x, y) plane. One of these constants of the motion is $v_c = \sqrt{v_x^2 + v_y^2}$, the **circular velocity**. Integrating once more,

$$x = x_0 + r_g\sin(\omega t + \alpha) \quad , \quad y = y_0 + r_g\cos(\omega t + \alpha) \quad , \tag{49}$$

where we have resolved the complex exponential into real form. Here,

$$r_g \equiv \frac{v_c}{\omega} = \frac{v_c\mathcal{E}}{qBc} = \frac{p_c c}{qB} \tag{50}$$

is called the **gyroradius** and is the radius of the circular projection of the motion about the magnetic field direction. Also, here $p_c = v_c\mathcal{E}/c^2$ is the

circular momentum. The solution of the z-direction component of the EOM is almost trivial:

$$z \;=\; z + v_z t \quad . \tag{51}$$

The complete trajectory defined by Eqs. (49) and (51) is a **helix**. The helical path is stretched or compressed according to the value of v_z/v_c, and so one can determine

$$\theta_\mu \;=\; \arctan\frac{v_z}{v_c} \tag{52}$$

to be the **pitch angle** of the helix. The motion possesses a characteristic angular frequency ω of rotation of the tip of the velocity vector about **B**, and we term this angular rate the relativistic **gyrofrequency**. We can write

$$\omega \;=\; \frac{qcB}{\mathcal{E}} \;=\; \frac{\omega_{\rm B}}{\gamma} \quad \text{with} \quad \omega_{\rm B} \;=\; \frac{qB}{mc} \tag{53}$$

for $\mathcal{E} = \gamma mc^2$. The intrinsically non-relativistic quantity $\omega_{\rm B}$ is called the **cyclotron frequency** and defines the natural scale of radiation by non-relativistic charges in magnetic fields.

Plot: Draw helix including pitch angle

- The presence of the $1/\gamma$ factor in ω is a relativistic time-dilation factor in boosting from an instantaneous stationary frame for the charge.

- If one constructs a non-relativistic simple harmonic oscillator in a uniform magnetic field, if its natural frequency is ω_0 (obtained from the restoring force), then the vector equation of motion is **L&L Sec. 21 Problem**

$$m\,\frac{d\mathbf{v}}{dt} \;=\; \frac{q}{c}\,\mathbf{v}\times\mathbf{B} - m\,\omega_0^2\,\mathbf{r} \quad . \tag{54}$$

This can be solved to yield sinusoidal components in all three dimensions. In the z-direction, the field exerts no force and the natural frequency ω_0 is reproduced. In the (x, y) plane, the *frequency is split*:

$$\omega \;=\; \sqrt{\omega_0^2 + \left(\frac{qB}{2mc}\right)^2} \pm \frac{qB}{2mc} \;\approx\; \omega_0 \pm \frac{\omega_{\rm B}}{2} \quad , \tag{55}$$

where the approximation is in the low field, $\omega_{\rm B} \ll \omega_0$ limit. This serves to define a classical model for **Zeeman splitting** of atomic lines.

3.3 Motion in Uniform Electric and Magnetic Fields

We will consider two specialized cases to illustrate a representative range of possibilities. For the general electromagnetic field, as we shall soon see, Lorentz transformation can *usually* be used to eliminate either an electric field (when $E < B$) or a magnetic field (when $B < E$), so that the preceding exclusive solutions have quite general applicability. An *exception is when the two fields are parallel to each other.* This will motivate our first case. **L&L Sec. 22**

• The book offers a problem on motion in crossed electromagnetic fields, where $\mathbf{E} \perp \mathbf{B}$ and $E = B$. This can be quite applicable to problems involving laser fields, such as motions of plasmas in **hohlraums** at the Omega or National Ignition Facility (NIF). We will not explore such a case here.

3.3.1 Parallel Electric and Magnetic Fields

• A motivation for this case is that when $\mathbf{E} \parallel \mathbf{B}$, *no Lorentz boost can eliminate either field entirely.* We will see this in due course, but the proof is that any boost can be distilled into the combination $\Lambda^{\alpha}_{\beta}\Lambda'^{\mu}_{\nu}$ of sequential boosts where one is parallel to the fields and the other perpendicular. The first does not alter the directions of the fields. The second introduces components of both the **E** and **B** fields that are orthogonal to the boost vector and non-zero. Thus the $\mathbf{E} \parallel \mathbf{B}$ case is an important, distinct situation.

Let the z-axis be the common direction of **E** and **B**. Only the electric field does work as a charge moves in this direction; gyrational motion in the transverse directions, the (x, y) plane, do not increase the particle's energy. This fact simplifies the solution of the Lorentz force equation, which is

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E} + \frac{q}{c}\mathbf{v} \times \mathbf{B} \quad . \tag{56}$$

Isolating the z-component, since the magnetic contribution for such is zero, we form an ODE for p_z and simply solve:

$$\frac{dp_z}{dt} = qE \quad \Rightarrow \quad p_z \equiv mc\gamma\beta_z = qEt \quad , \tag{57}$$

presuming that $p_z = 0$ initially. As this stands, this cannot be further integrated to solve for $z(t)$ without information on the energy or $\gamma(t)$. To acquire this, form the virial dot product of $\mathbf{p}$ with Eq. (56):

$$\frac{1}{2}\frac{d}{dt}\left(\frac{\mathcal{E}_{\rm K}^2}{c^2}\right) \;=\; \mathbf{p}\cdot\frac{d\mathbf{p}}{dt} \;=\; q\mathbf{p}\cdot\mathbf{E} \;=\; qE\,p_z \;=\; (qE)^2 t \quad . \tag{58}$$

Here we have again introduced the **kinematic energy** of the particle, $\mathcal{E}_{\rm K} = \sqrt{m^2c^4 + p^2c^2}$. The energy solution is therefore simply obtained, and is just as for the pure uniform $\mathbf{E}$ case:

$$\mathcal{E}_{\rm K} \;\equiv\; \gamma mc^2 \;=\; \sqrt{m^2c^4 + p_c^2c^2 + (qEct)^2} \;=\; \sqrt{\mathcal{E}_0^2 + (qEct)^2} \quad , \tag{59}$$

where $\mathcal{E}_0$ is the value of $\mathcal{E}_{\rm K}$ at $t = 0$, when the total momentum is just the circular momentum p_c. Since the time dependence of the energy is captured purely via the z-motion, i.e., within p_z, it follows that *the circular momentum is a conserved quantity.* The motion in the z-direction is now fully determined via the work equation:

$$z \;=\; \frac{\mathcal{E}_{\rm K} - \mathcal{E}_0}{qE} \quad\Rightarrow\quad \frac{v_z}{c} \;\equiv\; \beta_z \;=\; \frac{qEct}{\mathcal{E}_{\rm K}} \quad , \tag{60}$$

which is, of course, consistent with Eq. (57). If we set $\tau = qEct/\mathcal{E}_0$, then

$$z \;=\; \frac{\mathcal{E}_0}{qE}\left\{\sqrt{1+\tau^2} - 1\right\} \quad \text{and} \quad \beta_z \;=\; \frac{\tau}{\sqrt{1+\tau^2}} \tag{61}$$

are the functions of this dimensionless time variable.

The transverse motion solves in a manner similar to the uniform magnetic field example. However, since the energy is now time-dependent, solving for momenta or speeds in terms of time is a little more involved. Form

$$\frac{dp_x}{dt} \;=\; \frac{qB}{c}\,v_y \;=\; m\omega_{\rm B}\,v_y \quad , \quad \frac{dp_y}{dt} \;=\; -\frac{qB}{c}\,v_x \;=\; -m\omega_{\rm B}\,v_x \quad . \tag{62}$$

Here, the cyclotron frequency $\omega_{\rm B} = qB/mc$ has been introduced. Now combine these in complex form to define an ODE for $p_x + ip_y \equiv p_c e^{-i\phi}$, i.e.

$$\frac{d}{dt}\left(p_c e^{-i\phi}\right) \;\equiv\; \frac{d}{dt}\left(p_x + ip_y\right) \;=\; -i\omega_{\rm B}\,\frac{mc^2}{\mathcal{E}_{\rm K}}\left(p_x + ip_y\right) \quad . \tag{63}$$

Since p_c is conserved, only the momentum phase $\phi(t)$ is a function of time. The new ODE is then

$$\frac{d\phi}{dt} \;=\; \frac{\omega_{\rm B} mc^2}{\mathcal{E}_{\rm K}} \;=\; \frac{\omega_{\rm B} mc^2}{\sqrt{\mathcal{E}_0^2 + (qEct)^2}} \quad . \tag{64}$$

Hence, the angular velocity monotonically decreases with time. The solution for the momentum gyrophase involves hyperbolic functions:

$$\phi \;=\; \frac{B}{E}\sinh^{-1}\tau \;\equiv\; \frac{B}{E}\log_e\Bigl[\tau + \sqrt{1+\tau^2}\,\Bigr] \quad , \quad \tau \;=\; \frac{qEct}{\mathcal{E}_0} \quad . \tag{65}$$

Clearly the $E \to 0$ limiting form reproduces the purely magnetic case, with $d\phi/dt = \omega_{\rm B}/\gamma$. To solve completely for the transverse motion, Eq. (63) can be recast using $\mathbf{p} = \mathbf{v}\mathcal{E}_{\rm K}/c^2$ as

$$-ip_c e^{-i\phi}\frac{d\phi}{dt} \;=\; -i\omega_{\rm B}\frac{mc^2}{\mathcal{E}_{\rm K}}(p_x + ip_y) \;=\; -im\omega_{\rm B}\frac{d}{dt}(x+iy) \quad , \tag{66}$$

which is effectively an ODE in the variable ϕ. Thus for an initial position of $x = x_0$ and $y = y_0$,

$$\frac{d}{d\phi}(x+iy) \;=\; \frac{p_c}{m\omega_{\rm B}}\,e^{-i\phi} \quad \Rightarrow \quad x+iy \;=\; \frac{ip_c}{m\omega_{\rm B}}\,e^{-i\phi} + \text{const} \,. \tag{67}$$

Resolving this into real functions, the (x,y)-plane solution is therefore

$$x \;=\; \frac{p_c c}{qB}\sin\phi + x_0 \quad , \quad y \;=\; \frac{p_c c}{qB}\bigl(\cos\phi - 1\bigr) + y_0 \quad , \tag{68}$$

with a gyroradius $r_g = p_c c/qB$ that is independent of time. The remaining piece is the z-motion, which can be written using Eqs. (61) and (65) as

$$z \;=\; \frac{\mathcal{E}_0}{qE}\left\{\cosh\left(\frac{E}{B}\,\phi\right) - 1\right\} \quad . \tag{69}$$

The trajectory is thus a progressively-stretched helix with a monotonically increasing pitch angle. The motion has an asymptotic speed of c at infinity.

• Observe that the trajectory and solutions for this $\mathbf{E} \parallel \mathbf{B}$ case possess elements of both the uniform $\mathbf{E}$ field and uniform $\mathbf{B}$ field examples.

3.3.2 Oblique Electric and Magnetic Fields

To generalize again, we now introduce a component of the electric field that is not parallel to the magnetic field, i.e. $\mathbf{E} \times \mathbf{B} \neq \mathbf{0}$. To keep the algebra simple, we will restrict considerations to <u>non-relativistic</u> charges.

- The magnetic field will be in the z direction, and without loss of generality, the electric field will be in the (y, z)-plane. Thus the Lorentz force EOM is

$$m\,\frac{d\mathbf{v}}{dt} \;=\; q\mathbf{E} + \frac{q}{c}\,\mathbf{v} \times \mathbf{B} \quad , \tag{70}$$

and the component equations are

$$m\ddot{x} \;=\; \frac{q}{c}\,\dot{y}\,B \quad , \quad m\ddot{y} \;=\; qE_y - \frac{q}{c}\,\dot{x}\,B \quad , \quad m\ddot{z} \;=\; qE_z \quad . \tag{71}$$

The z-component of the motion is the non-relativistic limit of that in the $\mathbf{E} \parallel \mathbf{B}$ example, i.e. is quadratic in time:

$$z \;=\; v_{0z}t + \frac{qE_z}{2m}\,t^2 \quad . \tag{72}$$

For the motion transverse to $\mathbf{B}$, we use the same complex variable approach as before, combining the two dimensions:

$$\frac{d}{dt}\,(\dot{x} + i\dot{y}) + i\omega_{\rm B}\,(\dot{x} + i\dot{y}) \;=\; i\,\frac{qE_y}{m} \quad . \tag{73}$$

Again, $\omega_{\rm B} = qB/mc$ is the cyclotron frequency. This ODE is now inhomogeneous, and so the *complimentary function* is of the form $a\exp\{-i\omega_{\rm B}t\}$ as before. The *particular integral* is easy to solve for, and is simply a constant that must be real. Thus, we find, in real form

$$\dot{x} \;=\; v_c \cos\omega_{\rm B}t + \frac{cE_y}{B} \quad , \quad \dot{y} \;=\; -v_c \sin\omega_{\rm B}t \tag{74}$$

for the case where the velocity is initially in the (x, z) plane, a case with representative generality. The constant of integration, the circular velocity v_c, is determined by the initial conditions. For this solution to remain valid,

$$E_\perp \;\equiv\; E_y \;\ll\; B \tag{75}$$

defines the *non-relativistic criterion*. This is a condition that is independent of Lorentz boosts to other frames of reference. Moreover it is a fundamental to Larmor formalism for radiation, as we will discover in due course.

• The motion in velocity space is clearly oscillatory, and the average motion exhibits a **drift velocity** orthogonal to both **E** and **B**:

$$\mathbf{v}_{\rm D} \;=\; \frac{c\,\mathbf{E}\times\mathbf{B}}{B^2} \quad\Rightarrow\quad \langle \dot{x}\rangle \;=\; \frac{cE_\perp}{B} \quad , \quad \langle \dot{y}\rangle \;=\; 0 \quad . \tag{76}$$

Observe that the drift speed is independent of *both* the charge and the mass of the particle. Thus, *such drifts do not polarize a plasma*, i.e. electrons, ions, and positrons all move together.

∗ Rotating field configurations such as for spinning stars induce such drift velocities, as do Lorentz boosts between inertial frames.

The velocity equations can easily be integrated, and if we choose for the charge to be initially at the origin, then we obtain

$$x \;=\; \frac{v_c}{\omega_{\rm B}}\biggl(\sin\omega_{\rm B}t + \frac{c}{v_c}\frac{E_\perp}{B}\,\omega_{\rm B}t\biggr) \quad , \quad y \;=\; \frac{v_c}{\omega_{\rm B}}\Bigl(\cos\omega_{\rm B}t - 1\Bigr) \quad . \tag{77}$$

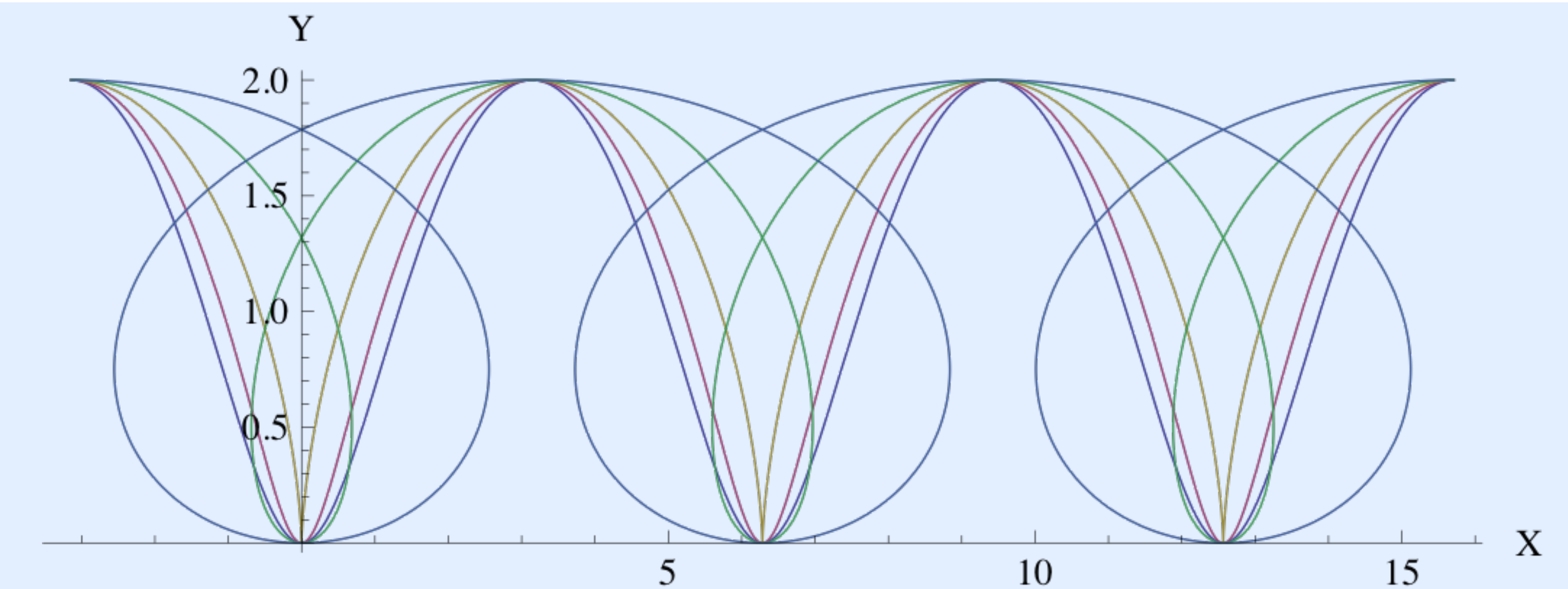
This class of curves for the cross section of the trajectory in the plane orthogonal to **B** is known as a set of **trochoids**. Their shape is determined by the parameter $\Lambda = v_c B/cE_\perp$: see the Figure. In the special case of $\Lambda = \pm 1$,

$$x \;=\; \frac{v_c}{\omega_{\rm B}}\biggl(\sin\omega_{\rm B}t \pm \omega_{\rm B}t\biggr) \quad , \quad y \;=\; \frac{v_c}{\omega_{\rm B}}\Bigl(\cos\omega_{\rm B}t - 1\Bigr) \quad . \tag{78}$$

and the trochoid reduces to a **cycloid**.

Plot: Trochoid sections for charges moving in oblique **E** and **B** fields.

Trochoid Sections for Trajectories of Charges in Oblique E and B Fields



Representative **trochoid** cross sections of the trajectories of a charge in oblique **E** and **B** fields, given by the parametric mathematical form

$$x = \tau - \Lambda \sin\tau \quad , \quad y = 1 - \cos\tau \quad .$$

These sections are in the plane orthogonal to **B**. Here $\tau = \omega_{\rm B} t$, and the parameter $\Lambda = v_c B/(cE_\perp)$ takes on values $0.25, 0.5, 1, 2$ and 4. The $\Lambda = 1$ case defines a **cycloid** with its characteristic cusps, and when $\Lambda > 1$, the paths exhibit **retrograde** motion; for $\Lambda < 1$ they are quasi-sinusoidal.