• The Lorentz force equation evinces the property that time reversal of the motion, with $t \to -t$ and $\mathbf{v} \to -\mathbf{v}$ (i.e. sending the particle backwards), one needs the electric field to remain unchanged, but the magnetic field must reverse its sign:

$$t \;\to\; -t \quad \Rightarrow \quad \mathbf{E} \to \mathbf{E} \quad , \quad \mathbf{B} \;\to\; -\mathbf{B} \quad . \tag{19}$$

Thus, as intimated in our exploration of tensors,

$$\begin{aligned} &\mathbf{E} \quad \text{is a polar vector,} \\ &\mathbf{B} \quad \text{is an axial vector.} \end{aligned}$$

This construction determines their suitable placement in any electromagnetic field tensor. In terms of the four-potential, this is then a transformation

$$t \;\to\; -t \quad \Rightarrow \quad \phi \to \phi \quad , \quad \mathbf{A} \;\to\; -\mathbf{A} \quad ; \tag{20}$$

the vector potential changes sign, but the scalar potential does not.

• Changing the sign of the charge $e \to -e$ leads to **helicity reversal** of trajectories, i.e. is reflection-symmetric in velocity space: this is used to effect for **charge discrimination** in particle physics experiments (LHC, RHIC, FermiLab) and cosmic ray detectors such as AMS.

Plot: Anti-particle Trajectories in a Magnetized Bubble Chanber

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Alternate derivation

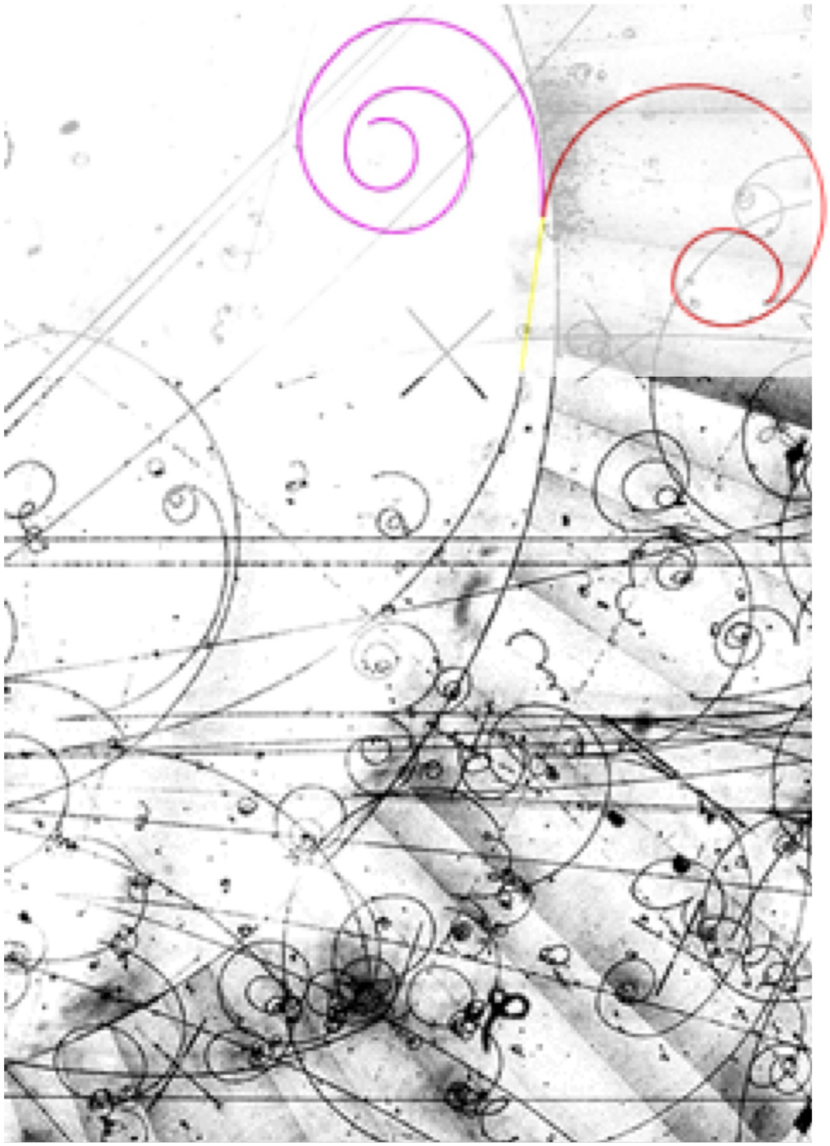
The equation motion can be derived instead via fully covariant development. In this the starting point is the covariant form of the Euler-Lagrange equations combined with a manifestly covariant Lagrangian $\mathcal{L}$. Thus

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial u^\mu}\right) \;=\; \frac{\partial \mathcal{L}}{\partial x^\mu} \quad \text{with} \quad \mathcal{L} \;=\; mc^2\sqrt{u_\nu u^\nu} + q\, A_\nu u^\nu \quad . \tag{21}$$

Here the four velocity $u^\nu = dx^\nu/ds$ has no explicit (only implicit) dependence on coordinates x^ν, but the four-potential $A_\nu(x^\nu)$ does. Accordingly, we set $\mathcal{L} = \mathcal{L}(x^\nu,\, u^\nu)$ when taking the various partial derivatives. First,

$$\frac{\partial \mathcal{L}}{\partial x^\mu} \;=\; q\,\frac{\partial A_\nu}{\partial x^\mu}\, u^\nu \quad . \tag{22}$$

e^+e^- Pairs in a Bubble Chamber



- Tracks in a ***FermiLab*** experiment bubble chamber, highlighting an electron (red)-positron (purple) pair produced by a photon (yellow).
- The photon was produced by a charged particle colliding with a nucleus.
- A ***B*** field generates the curvature of charge trajectories, which terminate due to energy loss.
- The direction of the curves highlights the sign of the charge.
- *Credit*: from the CERN HST2005 on-line archive.

The four-velocity derivative is more involved in that the metric tensor must be introduced in order to render the kinematic portion of the Lagrangian in a form that only involves only contravariant forms of the four-velocity:

$$
\begin{aligned}
\frac{\partial}{\partial u^\mu}\sqrt{u_\nu u^\nu} \;&\equiv\; \frac{\partial}{\partial u^\mu}\sqrt{g_{\nu\rho}u^\rho u^\nu} \;=\; \frac{1}{2}\,\frac{1}{\cancel{\sqrt{u_\nu u^\nu}}^{\,1}}\,\frac{\partial}{\partial u^\mu}\Bigl(g_{\nu\rho}u^\rho u^\nu\Bigr)\\
&=\; \frac{1}{2}\Bigl(g_{\nu\rho}\delta^\rho_\mu u^\nu + g_{\nu\rho}u^\rho\delta^\nu_\mu\Bigr) \;=\; u_\mu \quad .
\end{aligned}
\tag{23}
$$

The other part of the four-velocity derivative is trivial, so that the covariant form of the **canonical four-momentum** can be written

$$
P_\mu \;\equiv\; \frac{1}{c}\,\frac{\partial\mathcal{L}}{\partial u^\mu} \;=\; mc\,u_\mu + \frac{q}{c}\,A_\mu \quad , \tag{24}
$$

so that the contravariant form is

$$
P^\mu \;=\; \left(\gamma mc + \frac{q\phi}{c},\, \mathbf{p} + \frac{q}{c}\,\mathbf{A}\right) \quad . \tag{25}
$$

- We now have all the requisite pieces to manipulate the LHS of the Euler-Lagrange equations. Thus,

$$
\frac{d}{ds}\left(\frac{\partial\mathcal{L}}{\partial u^\mu}\right) \;=\; mc^2\,\frac{du_\mu}{ds} + q\,\frac{dA_\mu}{ds} \;=\; mc^2\,\frac{du_\mu}{ds} + q\,\frac{\partial A_\mu}{\partial x^\nu}\,u^\nu \quad , \tag{26}
$$

where we have employed the chain rule for differentiation on the term involving the four-potential $A_\nu(x^\nu)$. We then arrive at the covariant form for the Lorentz force equation of motion involving the **E/M field tensor** $F_{\mu\nu}$:

$$
mc^2\,\frac{du_\mu}{ds} \;=\; q\,F_{\mu\nu}u^\nu \quad \text{for} \quad F_{\mu\nu} \;\equiv\; \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \quad . \tag{27}
$$

This we will derive directly from the action in due course. Clearly, the tensor $F_{\mu\nu}$ is anti-symmetric. In particular, the $\mu = 0$ component yields the work done, so that one infers that the $F_{0\nu}$ components capture only $\mathbf{E}$ field information (polar vector).

2.1 Uniform Fields

Constant or uniform fields are useful constructs in experiments. They are such that *the field intensity is the same at all points in space*, at least within some volume of interest. Uniform $\mathbf{E}$ fields can be constructed using large planar sheets of charge, or capacitances. Uniform $\mathbf{B}$ fields can be approximated by running currents through tightly-wound coils of significant length: the interior is then a constant magnetic field to good approximation. **L&L Sec. 19**

Plot: Hand-drawn diagram of capacitance and coil.

For a uniform electric field, the potential function ϕ has a preferred direction:

$$\nabla\phi \;=\; -\mathbf{E} \quad \Rightarrow \quad \phi \;=\; -\mathbf{E}\cdot\mathbf{r} \quad , \tag{28}$$

where we note that since any spatial derivative of $\mathbf{E}$ is zero,

$$\nabla(\mathbf{E}\cdot\mathbf{r}) \;\equiv\; (\mathbf{E}\cdot\nabla)\mathbf{r} + \cancel{(\mathbf{r}\cdot\nabla)\mathbf{E}}^{\,0} + \mathbf{E}\times\cancel{(\nabla\times\mathbf{r})}^{\,0} + \mathbf{r}\times\cancel{(\nabla\times\mathbf{E})}^{\,0} \tag{29}$$

so that $\nabla(\mathbf{E}\cdot\mathbf{r}) = \mathbf{E}$. The vector potential in a uniform magnetic field also assumes a simple form:

$$\mathbf{B} \;=\; \nabla\times\mathbf{A} \quad \Rightarrow \quad \mathbf{A} \;=\; \frac{1}{2}\mathbf{B}\times\mathbf{r} \quad . \tag{30}$$

This is a little trickier to derive, but again we use a well-known formula from vector analysis:

$$\nabla\times(\mathbf{B}\times\mathbf{r}) \;=\; \mathbf{B}\,\nabla\cdot\mathbf{r} - (\mathbf{B}\cdot\nabla)\mathbf{r} \;=\; 2\,\mathbf{B} \quad , \tag{31}$$

noting that $\nabla\cdot\mathbf{r} = 3$ and that resolving each component yields $(\mathbf{B}\cdot\nabla)\mathbf{r} = \mathbf{B}$.

2.2 Gauge Invariance

Thus far the formulation has involved a quantity called the four-potential that has limited physical information, in that it cannot be measured in totality, just inferred. Since it appears only in terms of its gradients in the equations of motion, *it cannot be uniquely determined by a given physical system*, i.e. one set of values for $\mathbf{E}$ and $\mathbf{B}$ does not specify A_μ uniquely. **L&L Sec. 18**

Consider a **gauge transformation** of the four-potential:

$$A_\mu \;\to\; A'_\mu \;=\; A_\mu - \frac{\partial f}{\partial x^\mu} \tag{32}$$

for some arbitrary scalar function $f(x^\nu)$ of spacetime coordinates. Under this transformation, the matter-field contribution to the action changes thus:

$$S_{mf} \;\to\; \frac{q}{c}\int_a^b A_\mu \, dx^\mu - \frac{q}{c}\int_a^b \frac{\partial f}{\partial x^\mu}\, dx^\mu \quad . \tag{33}$$

It is simply observed that for each of the spacetime components, the second integral is a perfect derivative and thus is expressible in terms of the function $f(x^\nu)$ evaluated at the endpoints a and b of the spacetime path.

• Thus with the principle of least action, *this gauge transformation cannot change the equation of motion, so that the fields are left invariant.*

To elucidate this further, distilling the gauge transformation into its spacetime components, we have

$$\mathbf{A}' \;=\; \mathbf{A} + \nabla f \quad , \quad \phi' \;=\; \phi - \frac{1}{c}\frac{\partial f}{\partial t} \quad . \tag{34}$$

and it is clearly determined that such transformations do not alter the fields

$$\mathbf{E} \;=\; -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi \quad \text{and} \quad \mathbf{B} \;=\; \nabla\times\mathbf{A} \quad . \tag{35}$$

This establishes the principle of the **gauge invariance** of electromagnetic fields. Common examples include the **Coulomb gauge**, where $\nabla\cdot\mathbf{A} = 0$, the **Lorenz gauge**, where $\nabla\cdot\mathbf{A} = -(1/c)\,\partial\phi/\partial t$ or $\partial^\mu A_\mu = 0$ in covariant form. Less popular is the covariant Fock-Schwinger gauge, $x^\mu A_\mu = 0$.

• In due course, we shall discern that motions are not impacted by gauge transformations, <u>an invariance that is connected to charge conservation</u>.

• Observe that *neither the Lagrangian nor the Hamiltonian are gauge invariant* since they explicitly involve ϕ and $\mathbf{A}$. This is not a problem as they are functional forms, and are not experimentally determined; all measured energies are obtained via differences (i.e., work done) that do not calibrate a ground state energy, which can be absorbed in a choice of gauge.

3 Charge Motions in Electromagnetic Fields

Three cases of motions in static, uniform electromagnetic fields are explored here, to capture the essence of particle mechanics. The examples are for a uniform $\mathbf{E}$ field, a uniform $\mathbf{B}$ field, an a combination of the two. The summary results are:

- in a uniform $\mathbf{E}$ field, the trajectory is a planar catenary curve;
- in a uniform $\mathbf{B}$ field, the trajectory is a helix,
- in parallel $\mathbf{E}$ and $\mathbf{B}$ fields, the path is a progressively stretched helix,
- in oblique $\mathbf{E}$ and $\mathbf{B}$ fields, the path has a cross section of a cycloid or a trochoid perpendicular to **B**, stretched by acceleration parallel to **B**.

3.1 Motion in Uniform Electric Fields

Assume that $\mathbf{E} = E_x \hat{x}$. The motion is obviously in a plane, which we will presume to be in the $x-y$ plane. If the initial component of speed in the y-direction is zero, then the motion will be rectilinear. The Lorentz force equation of motion yields **L&L Sec. 20**

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E} \quad \Rightarrow \quad \dot{p}_x = qE \quad , \quad \dot{p}_y = 0 \quad , \tag{36}$$

for $E_x \equiv E$. We will presume truly relativistic particles so that $\mathbf{p} = \gamma m \mathbf{v}$. The integrals of the momentum components are

$$p_x = qE\,t \quad , \quad p_y = p_0 \quad . \tag{37}$$

Here we assume that $p_x = 0$ at $t = 0$, though this can be easily generalized. The **kinematic energy** of the particle is $\mathcal{E}_{\rm K} = \sqrt{m^2c^4 + p^2c^2}$, and represents the Hamiltonian without the contribution from the electrostatic potential ϕ. Note that we distinguish this from the **kinetic energy** $\mathcal{E}_{\rm K} - mc^2$, and remark that the standard terminology here differs from that in L&L. Therefore,

$$\mathcal{E}_{\rm K} = \sqrt{m^2c^4 + p_0^2c^2 + (qEct)^2} = \sqrt{\mathcal{E}_0^2 + (qEct)^2} \quad , \tag{38}$$

where $\mathcal{E}_0$ is the value of $\mathcal{E}_{\rm K}$ at $t = 0$.

The velocity of the charge is $\mathbf{v} = \mathbf{p}c^2/\mathcal{E}_{\mathrm{K}}$. From this one forms the differential equation for the evolution of the component of velocity in the x-direction, $v_x = \beta_x c$. Using Eq. (37), we have

$$\beta_x \; = \; \frac{1}{c}\,\frac{dx}{dt} \; = \; \frac{p_x c}{\mathcal{E}_{\mathrm{K}}} \; = \; \frac{qE\,ct}{\sqrt{\mathcal{E}_0^2 + (qEct)^2}} \quad . \tag{39}$$

If we scale the time variable via $\tau = qE\,ct/\mathcal{E}_0$, this is routinely integrated:

$$\frac{dx}{d\tau} \; = \; \frac{\mathcal{E}_0}{qE}\,\frac{\tau}{\sqrt{1+\tau^2}} \quad \Rightarrow \quad x \; = \; \frac{\mathcal{E}_0}{qE}\left\{\sqrt{1+\tau^2} - 1\right\} \quad . \tag{40}$$

Here it has been assumed that $x = 0$ at $t = 0$ to determine the constant of integration.

The motion in the y-direction is defined by

$$\beta_y \; = \; \frac{1}{c}\,\frac{dy}{dt} \; = \; \frac{p_y c}{\mathcal{E}_{\mathrm{K}}} \; = \; \frac{p_0 c}{\sqrt{\mathcal{E}_0^2 + (qEct)^2}} \; = \; \frac{p_0 c}{\mathcal{E}_0}\,\frac{1}{\sqrt{1+\tau^2}} \quad , \tag{41}$$

which integrates to

$$y \; = \; \frac{p_0 c}{qE}\,\sinh^{-1}\tau \; \equiv \; \frac{p_0 c}{qE}\,\log_e\!\left[\tau + \sqrt{1+\tau^2}\,\right] \quad . \tag{42}$$

If we use this to eliminate τ in favor of y, and substitute in Eq. (40), the result is

$$\frac{qE}{\mathcal{E}_0}\,x \; = \; \cosh\left(\frac{qE}{p_0 c}\,y\right) - 1 \quad . \tag{43}$$

Thus *the general path for constant acceleration in special relativity*, in this case precipitated by a uniform electric field, as a **catenary curve**.

• In the special case of non-relativistic motions with $v \ll c$, we can set $p_0 = mv_0$ and $\mathcal{E}_0 = mc^2$, and expand the solution in powers of v/c. For the trajectory shape, this is tantamount to setting $qEy/p_0 c \ll 1$, and the leading order contribution from the Taylor series expansion of Eq. (43) yields

$$x \; \approx \; \frac{qE}{2mv_0^2}\,y^2 \quad . \tag{44}$$

This is a **parabolic path**, *the well-known result for the general path for constant acceleration in Galilean relativity*, i.e. classical mechanics.