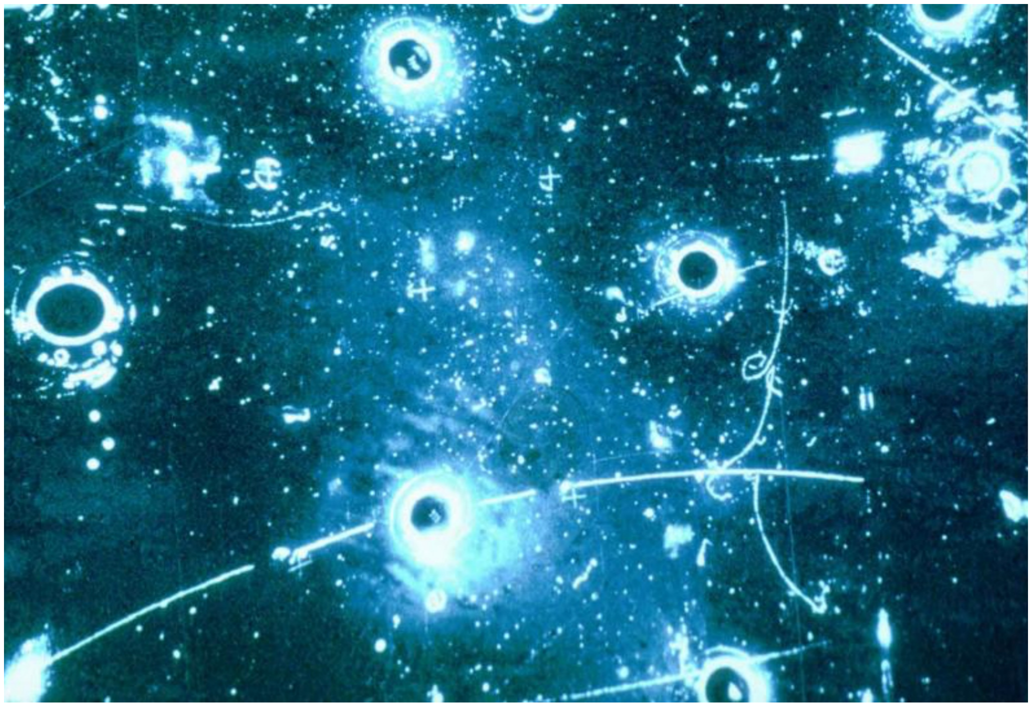
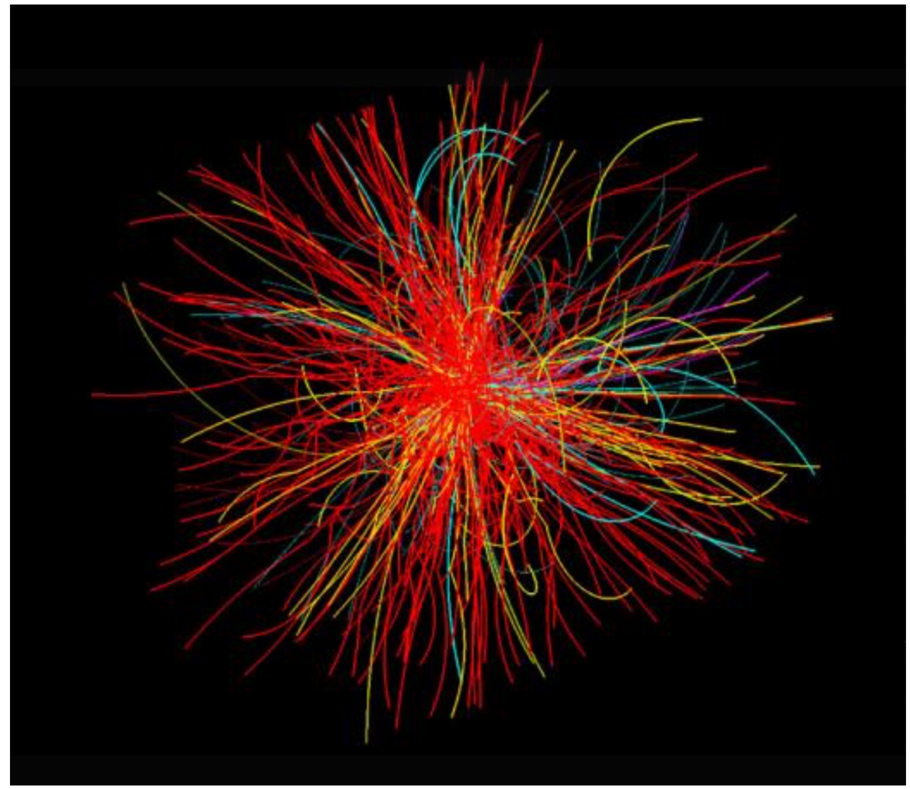
Tracks in Particle Physics Detector Chambers





- *Left*: tracks created by a neutrino entering from the top in CERN's Gargamelle bubble chamber (July 1973), probing weak neutral currents.
- *Right*: tracks precipitated in collisions of protons at 13 TeV CM frame energy (June 2015) in the LHC's ALICE detector. Courtesy: CERN News archive.

3.3 Kinematics of Relativistic Scattering

Many interactions in particle accelerators and in the cosmos involve scatterings of relativistic particles. We briefly outline how to assess the kinematics of such events. The exposition in L&L involves a certain amount of algebra that is not particularly enlightening, so our summary here will be limited.

Plot: Draw schematic of two colliding species

Considerations will first be restricted to elastic collisions of two particles, not necessarily identical. They suffice to illustrate the construction that can extend to conversion of species. We will use subscripts f to denote final species (instead of primes), and leave off subscripts for the ingoing particles. Four-momentum conservation gives **L&L Sec. 13**

$$p_1^\mu + p_2^\mu \;=\; p_{1f}^\mu + p_{2f}^\mu \quad . \tag{57}$$

This can be re-arranged and squared in different ways to develop energy and momentum conservation algebra. We give an example here to illustrate the protocols. First, noting that $p_\mu p^\mu = m^2$ (hereafter, we will set $c \to 1$), form

$$\Bigl(p_1^\mu + p_2^\mu - p_{1f}^\mu\Bigr)^2 \;=\; \bigl(p_{2f}^\mu\bigr)^2 \quad\Rightarrow\quad m_1^2 + p_{1\mu} p_2^\mu - \Bigl(p_{1\mu} + p_{2\mu}\Bigr) p_{1f}^\mu \;=\; 0\ . \tag{58}$$

This has divided the square by two. A similar relation can be formed by interchanging the particles:

$$\Bigl(p_1^\mu + p_2^\mu - p_{2f}^\mu\Bigr)^2 \;=\; \bigl(p_{1f}^\mu\bigr)^2 \quad\Rightarrow\quad m_2^2 + p_{1\mu} p_2^\mu - \Bigl(p_{1\mu} + p_{2\mu}\Bigr) p_{2f}^\mu \;=\; 0\ . \tag{59}$$

Each of the four-momentum products in these equations can be evaluated for a specific reference frame. The preferred choices are the rest frame (L-frame) of one of the species, and the center-of-momentum (CM) frame. The sum and difference between these two identities are

$$\begin{aligned} m_1^2 - m_2^2 \;&=\; \Bigl(p_{1\mu} + p_{2\mu}\Bigr)\Bigl(p_{1f}^\mu - p_{2f}^\mu\Bigr) \quad , \\ m_1^2 + m_2^2 + 2\, p_{1\mu} p_2^\mu \;&=\; \Bigl(p_{1\mu} + p_{2\mu}\Bigr)\Bigl(p_{1f}^\mu + p_{2f}^\mu\Bigr) \quad . \end{aligned} \tag{60}$$

The second of these, the sum of the two identities, is trivial if Eq. (57) is used to replace the final four-momenta.

• For an L-frame case, we set m_2 at rest initially. Then $p_2^\mu = (m_2, \mathbf{0})$. Now let βc be the speeds of the particles (incoming or outgoing) in the L-frame. Then the four-momentum products are

$$\begin{aligned}
p_{1\mu}p_2^\mu &= \mathcal{E}_1 m_2 \;\rightarrow\; \gamma_1\, m_1 m_2 \\
p_{1\mu}p_{1f}^\mu &= \mathcal{E}_1\mathcal{E}_{1f} - \mathbf{p}_1\cdot\mathbf{p}_{1f} \;\rightarrow\; \gamma_1\gamma_{1f}\Big(1-\beta_1\beta_{1f}\cos\theta_{1f}\Big)\, m_1^2 \\
p_{1\mu}p_{2f}^\mu &= \mathcal{E}_1\mathcal{E}_{2f} - \mathbf{p}_1\cdot\mathbf{p}_{2f} \;\rightarrow\; \gamma_1\gamma_{2f}\Big(1-\beta_1\beta_{2f}\cos\theta_{2f}\Big)\, m_1 m_2 \qquad (61)\\
p_{2\mu}p_{1f}^\mu &= \mathcal{E}_{1f} m_2 \;\rightarrow\; \gamma_{1f} m_1 m_2 \\
p_{2\mu}p_{2f}^\mu &= \mathcal{E}_{2f} m_2 \;\rightarrow\; \gamma_{2f}\, m_2^2 \quad .
\end{aligned}$$

Here the θ_{1f} and θ_{2f} angles define the directions of the outgoing momenta relative to the incoming momentum $\mathbf{p}_1$. These angles are correlated with the energies of these final particles, and the precise relations can be determined by substitution of the appropriate evaluations into the 4-momentum conservation relations. One example is

$$\begin{aligned}
\gamma_{2f} &= \frac{(\mathcal{E}_1 + m_2c^2)^2 + p_1^2c^2\,\cos^2\theta_{2f}}{(\mathcal{E}_1 + m_2c^2)^2 - p_1^2c^2\,\cos^2\theta_{2f}} \\
&= \frac{(\gamma_1 m_1 + m_2)^2 + \gamma_1^2\beta_1^2 m_1^2\,\cos^2\theta_{2f}}{(\gamma_1 m_1 + m_2)^2 - \gamma_1^2\beta_1^2 m_1^2\,\cos^2\theta_{2f}}
\end{aligned} \qquad (62)$$

the proof of which is left as an exercise. Clearly, $\theta_{2f} = \pi/2$ defines a minimum for γ_{2f} ($= 1$, rest condition), while $\theta_{2f} = 0, \pi$ define maxima; this can be simply demonstrated graphically, or by direct differentiation.

The expression for γ_{1f} is more involved and will not be displayed here. Yet, for the massless case $m_1 \rightarrow 0$, e.g. for photons colliding with electrons at rest, one can quickly develop Eq. (58) along the lines of

$$\begin{aligned}
p_{1\mu}p_2^\mu &= p_{1\mu}p_{1f}^\mu + p_{2\mu}p_{1f}^\mu \\
\Rightarrow \quad \mathcal{E}_1 m_2 &= \mathcal{E}_1\mathcal{E}_{1f}(1-\cos\theta_{1f}) + \mathcal{E}_{1f} m_2 \quad .
\end{aligned} \qquad (63)$$

The second equation can be re-arranged, and the c^2 factors re-introduced:

$$\frac{\mathcal{E}_{1f}}{\mathcal{E}_1} = \frac{m_2c^2}{m_2c^2 + \mathcal{E}_1(1-\cos\theta_{1f})} \quad \Rightarrow \quad \lambda_f - \lambda = \frac{h}{mc}\Big(1-\cos\theta_{1f}\Big)\,. \qquad (64)$$

This, of course, is the classic **Compton formula**, where we have expressed the photon energies in terms of wavelengths: $\mathcal{E}_1 \rightarrow hc/\lambda$ and $\mathcal{E}_{1f} \rightarrow hc/\lambda_f$.

3. CHARGES IN ELECTROMAGNETIC FIELDS

Matthew Baring — Lecture Notes for PHYS 532, Spring 2023

Preface: The approach to the formulation of electromagnetism by Landau & Lifshitz follows a formal path that employs the principle of special relativity at its outset. Start with an appropriate form for the action, derive the forms for the Lagrangian and Hamiltonian, and thereafter deduce logical definitions for the fields that are consistent with historical E/M experiments. **L&L Sec. 15**

• We will continue to presume that *particles that are subject to electromagnetic forces are points*, and not extended; this can be accommodated in quantum mechanics where particles constitute spacetime probability distributions of localization: relativity impacts the coordinate description.

• The electromagnetic interaction between two remote particles is not instantaneous. One particle establishes an E/M field that the other experiences at a later time and vice versa. *Causal connection constrains information conveyance from one particle to the other according to the signal speed*. Accordingly, **retarded potentials** will eventually appear in the formalism.

$*$ In a vacuum, c is the communication speed of information about changes in the electromagnetic field.

• An immediate consequence of this causality restriction is that the concept of a **macroscopic rigid body** *is an approximation that is inaccurate* when rectilinear or rotational motion approaches the speed of light. Deformations within the body will be retarded as the E/M force (and any other force) communicates to remote parts of it.

$*$ Likewise, rigid field structure is an imperfect concept, e.g. pulsars. Relativistic retardation of field structure occurs in accelerating systems.

1 Action, Lagrangian and Hamiltonian

The equations of motion in an electromagnetic field are to be derived using an upgrade of the action. Having formulated the action for free particles, the logical step is to introduce additional terms that are either scalar, vector or tensor in character. A scalar modification should reduce to the one already posited for relativistic systems. Moreover, we know that the E/M force has vector character. The need to go to tensor modifications of the action would have to match constraints from observations. As it turns out, a vector contribution to S from electromagnetism suffices to match experimental results. We thus posit the form **L&L Sec. 16**

$$S \;=\; S_m + S_{mf} \;=\; \int_a^b \left(-mc\, ds - \frac{q}{c}\, A_\mu\, dx^\mu \right) \;\equiv\; \int_{t_a}^{t_b} L\, dt \tag{1}$$

for the covariant action in the presence of E/M forces and fields. We know that the charge q serves as a fundamental scale for the interaction, and so it serves as a multiplicative constant. The scaling as $1/c$ is introduced for convenience of units for A_μ. The structure $A_\mu\, dx^\mu$ is mandated by the requirement that the action be a scalar in relativity. Thus

$$S_{mf} \;=\; -\frac{q}{c} \int_a^b A_\mu\, dx^\mu \tag{2}$$

defines the matter interaction term for the action in which a charge responds to an electromagnetic field. Note that L&L use e to represent general q.

The aim is to determine what A_μ is. It is called the **four-potential**, and its contravariant and covariant forms are composed of elements

$$A^\mu \;=\; (\phi,\, \mathbf{A}) \quad \text{and} \quad A_\mu \;=\; (\phi,\, -\mathbf{A}) \quad , \tag{3}$$

where ϕ is the **scalar potential**, and $\mathbf{A}$ is the **vector potential**. The action then distills into separate space and time integrations, and if we set $\mathbf{v} = d\mathbf{r}/dt$ as the velocity of a particle whose motion we are considering, then

$$\begin{aligned} S \;&=\; \int_a^b \left(-mc\, ds + \frac{q}{c}\, \mathbf{A} \cdot d\mathbf{r} - q\phi\, dt \right) \\ &=\; \int_{t_a}^{t_b} \left(-\frac{mc^2}{\gamma} + \frac{q}{c}\, \mathbf{A} \cdot \mathbf{v} - q\phi \right) dt \end{aligned} \tag{4}$$

From this, we directly infer the form of the *Lagrangian for electromagnetism plus relativstic mechanics* in pseudo-covariant form:

$$L \;=\; -\frac{mc^2}{\gamma} + \frac{q}{c}\,\mathbf{A}\cdot\mathbf{v} - q\phi \quad . \tag{5}$$

It then follows that the corresponding Hamiltonian is

$$H \;=\; \mathbf{v}\cdot\frac{\partial L}{\partial \mathbf{v}} - L \;=\; \gamma mc^2 + q\phi \quad . \tag{6}$$

Thus, $\mathbf{A}$ can do no work, but ϕ can.

- This formalism routinely maps over to the quantum domain with the simple operator substitution $\mathbf{p} \to -i\hbar\nabla$.

We can transform this Lagrangian to formulate it in truly covariant form. For such, when writing down an action, we want both space and time on an equal footing. This is most compactly done by setting $dx^\mu \to u^\mu ds$ in the expression for the action in Eq. (1). We then *define* the new Lagrangian as the integrand of an integral over ds , which is then automatically in covariant form, thereby introducing a time dilation factor of γ in the revised definition. Yet one more subtlety is important. The mechanical part must retain the same velocity structure as before, at least in clever disguise. We define the **fully covariant Lagrangian** of a charge in an electromagnetic field to be

$$\mathcal{L} \;=\; mc^2\sqrt{u_\mu u^\mu} + qA_\mu\, u^\mu \quad . \tag{7}$$

This leverages the trivial result $u_\mu u^\mu = 1$, but retains implicit four-velocity dependence when derivatives are formed. Clearly $\mathcal{L} = -\gamma L$, and the change of sign is immaterial to what follows. Without formal proof (coming shortly), we assert that covariant equations of motion are obtained using

$$\boxed{\frac{d}{ds}\left(\frac{\partial\mathcal{L}}{\partial u^\mu}\right) \;=\; \frac{\partial\mathcal{L}}{\partial x^\mu}} \quad , \tag{8}$$

which are the **fully covariant Euler-Lagrange equations**. A result of this form will become apparent when the electromagnetic field tensor $F^{\mu\nu}$ is addressed shortly; it will facilitate an elegant derivation of the form of $F^{\mu\nu}$.

2 Equations of Motion for Charges

Returning to our three-space formulation of the Euler-Lagrange equations, we will use them to derive the vector equation of motion for a charge in an electromagnetic field. for $\mathbf{q} \to \mathbf{r}$, we have **L&L Sec. 17**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) \;=\; \frac{\partial L}{\partial \mathbf{r}} \quad . \tag{9}$$

Using Eq. (5), we consider first the RHS with $\mathbf{A} = \mathbf{A}(\mathbf{r})$, $\phi = \phi(\mathbf{r})$ and $\mathbf{v}$ fixed at a particular point in spacetime. Then

$$\frac{\partial L}{\partial \mathbf{r}} \;=\; \nabla L \;=\; \frac{q}{c}\,\nabla(\mathbf{A}\cdot\mathbf{v}) - q\nabla\phi \quad . \tag{10}$$

Herein, all ∇ operators possess partial derivative character. The vector potential term is manipulated via a standard identity in vector analysis:

$$\nabla(\mathbf{A}\cdot\mathbf{v}) \;=\; \cancel{(\mathbf{A}\cdot\nabla)\mathbf{v}} + (\mathbf{v}\cdot\nabla)\mathbf{A} + \mathbf{v}\times(\nabla\times\mathbf{A}) + \cancel{\mathbf{A}\times(\nabla\times\mathbf{v})} \quad . \tag{11}$$

Since the velocity does depend on $\mathbf{r}$ *explicitly*, terms involving derivatives of it are zero. Thus, the RHS of the Euler-Lagrange construction becomes

$$\frac{d}{dt}\left(\mathbf{p} + \frac{q}{c}\mathbf{A}\right) \;=\; \frac{\partial L}{\partial \mathbf{r}} \;=\; \frac{q}{c}\Big\{(\mathbf{v}\cdot\nabla)\mathbf{A} + \mathbf{v}\times(\nabla\times\mathbf{A})\Big\} - q\nabla\phi \quad . \tag{12}$$

As in the absence of the E/M field, the derivative of the $-mc^2/\gamma$ term in the Lagrangian yields $\mathbf{p}$. Now, the total time derivative of the vector potential on the LHS consists of two contributions: (i) the explicit $\partial\mathbf{A}/\partial t$ part, and the implicit variation contained in the displacement $d\mathbf{r} = \mathbf{v}\,dt$ due to the charge's motion. This yields

$$\frac{d\mathbf{A}}{dt} \;=\; \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A} \quad . \tag{13}$$

This construction is termed a **convective derivative**, and is commonly seen in fluid dynamic theory. Cancellation of identical terms on each side now emerges and the result is

$$\frac{d\mathbf{p}}{dt} \;=\; -q\left\{\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi\right\} + \frac{q}{c}\,\mathbf{v}\times(\nabla\times\mathbf{A}) \quad . \tag{14}$$

This is the desired equation of motion for the charge.

The force consists of two parts on the RHS, the first being independent of the velocity of the particle, and proportional to the charge. The ratio of the two is *defined to be* the **electric field intensity**:

$$\mathbf{E} \; = \; -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi \quad . \tag{15}$$

The second part of the force is proportional to the velocity, but is always perpendicular to it. Thus we define the **magnetic field intensity**:

$$\mathbf{B} \; = \; \nabla \times \mathbf{A} \quad . \tag{16}$$

A general electromagnetic field is thus a superposition of an electric and a magnetic field. The equation of motion then assumes the form

$$\frac{d\mathbf{p}}{dt} \; = \; q\left\{\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right\} \quad , \tag{17}$$

which is termed the **Newton-Lorentz force equation**.

• It is immediately apparent that adding a constant to either ϕ or $\mathbf{A}$ will not alter the electric and magnetic fields. This provides a lead-in to the discussion of gauge transformations shortly.

The rate at which the energy of a charge changes in an electromagnetic field is simply determined. Forming a scalar product of the velocity with the force,

$$\frac{d\mathcal{E}}{dt} \; = \; \mathbf{v} \cdot \frac{d\mathbf{p}}{dt} \; = \; q\,\mathbf{v} \cdot \mathbf{E} \; \rightarrow \; \frac{d\mathcal{E}_{\mathrm{K}}}{dt} \quad . \tag{18}$$

Here $\mathcal{E}_{\mathrm{K}} = (\gamma - 1)mc^2$ is the kinetic energy. Observe that *a magnetic field does no work on a charge: only the electric field change its energy.* The work done over a space interval $d\mathbf{r}$ is thus $dW = q\,\mathbf{E} \cdot d\mathbf{r}$.