2.2 Energy-Momentum Distribution Functions

Having forged a cohesive formalism for coupling energy and momentum in relativity, for the purposes of determining interaction probabilities and rates for collisions and decay of relativistic species, we need to discuss distribution functions. These are also central to thermalization considerations, and are germane to relativistic Maxwell-Boltzmann distributions and also Planck spectra — they form a basis for extension to quantum theory (i.e. QED). **L&L Sec. 10**

The first distribution function to be discussed is the 3D momentum space distribution function $f(\mathbf{p})$, which when integrated gives the total number N of particles in a system:

$$N \; = \; \int f(\mathbf{p})\, d^3p \; \equiv \; \int f(\mathbf{p})\, dp_x dp_y dp_z \quad . \tag{33}$$

This number is clearly a Lorentz invariant. To ascertain the Lorentz boost dependence of the distribution, we leverage our considerations so far of spatial coordinates:

$$d^4x \; = \; cdt\, dx\, dy\, dz \; = \; \gamma c\, d\tau\, d\mathcal{V} \; = \; \text{Lorentz invariant} \quad , \tag{34}$$

where γ is the boost Lorentz factor between the K and K' frames, and we know that $d\mathcal{V}$ transforms as $1/\gamma$ in such a boost (length contraction). Therefore *d^4x is a Lorentz invariant.* So too is any four-volume element formed from a *bona fide* four-vector. Thus,

$$d^4p \; \equiv \; dp^0\, dp^1\, dp^2\, dp^3 \; = \; \frac{d\mathcal{E}}{c}\, dp_x\, dp_y\, dp_z \; = \; \text{Lorentz invariant} \quad . \tag{35}$$

Due to the normalization of the four-momentum vector, the integrations are not all independent, and so must be weighted by a delta function, which must capture energy/momentum conservation. The connection between covariant and non-covariant forms is made via manipulation of the δ function:

$$\int \delta\Bigl(p_\mu p^\mu - m^2c^2\Bigr)\, d^4p \;\to\; \frac{c}{2}\int \delta\Bigl(\frac{\mathcal{E}^2}{c^2} - p^2 - m^2c^2\Bigr)\, \frac{2\mathcal{E}\, d\mathcal{E}}{c^2}\, \frac{d^3p}{\mathcal{E}} \;\to\; \int \frac{c\, d^3p}{2\mathcal{E}} \; , \tag{36}$$

as the energy integration is now trivial. Since the LHS is manifestly invariant, so also is $d^3p/\mathcal{E}$ Lorentz invariant.

The 4D momentum distribution function can be defined and related to the 3D one:

$$N = \int f(p^\mu)\, d^4p \quad \text{with} \quad f(p^\mu) \equiv f(\mathbf{p})\, \frac{2\mathcal{E}}{c}\, \delta\Big(p_\mu p^\mu - m^2 c^2\Big) \tag{37}$$

in covariant form. The need for the $2\mathcal{E}/c$ factor in Eq. (37) is obvious, by virtue of Eq. (36).

If we consider two frames K and K' in which a particle is moving, then defining the K_0 frame as the one in which the particle is at rest, we can write down the energies and differential volume elements in the K and K' frames:

$$\begin{aligned} d\mathcal{V} &= \frac{d\mathcal{V}_0}{\gamma} \quad , \quad d\mathcal{V}' = \frac{d\mathcal{V}_0}{\gamma'} \\ \mathcal{E} &= \gamma\, mc^2 \quad , \quad \mathcal{E}' = \gamma'\, mc^2 \quad . \end{aligned} \tag{38}$$

Here $d\mathcal{V}_0$ is the proper volume element in the particle's rest frame, and $\gamma = 1/\sqrt{1 - V^2/c^2}$, etc. The energy boosts exactly compensate the length contraction. Accordingly, given the Lorentz invariance of $d^3p/\mathcal{E}$, we can assert that

$$d^3p\, d^3x \tag{39}$$

is a Lorentz invariant, as is $\mathcal{E} d\mathcal{V} = mc^2\, d\mathcal{V}_0$. This is a direct consequence of x and p being conjugate variables in Hamiltonian formalism. From

$$N = \int f(\mathbf{x},\, \mathbf{p})\, d^3p\, d^3x \tag{40}$$

it follows that the **phase space density** $f(\mathbf{x},\, \mathbf{p})$ is also a Lorentz invariant:

$$f'(\mathbf{x}',\, \mathbf{p}') = f(\mathbf{x},\, \mathbf{p}) \quad . \tag{41}$$

This is the quantity that is used by heliospheric space physicists when measuring solar wind particles with spacecraft detectors.

• As an example of the extrapolation of this formalism to quantum mechanics, an electromagnetic contribution to the Hamiltonian must scale, to lowest order, linearly with the E/B field. Consider a charge (e.g. electron) in a uniform magnetic field $\mathbf{B}$. The cyclotron frequency is $\omega_{\mathrm{B}} = eB/mc$. This yields a quantum energy scale $\hbar\omega_{\mathrm{B}} = e\hbar B/mc$ so that

$$eB\hbar c \sim (mc^2)^2 \;\sim\; (pc)^2 \qquad (42)$$

defines the square of an energy scale. Since the electromagnetic correction to a Lagrangian/Hamiltonian does not yield translational invariance perpendicular to $\mathbf{B}$, the two transverse momentum dimensions are quantized leading the Landau levels in energy, thereby introducing a quantum number n. Thus

$$p_{\perp} \;\sim\; \sqrt{\frac{eB\hbar}{c}} \quad \Rightarrow \quad d^3p \;\equiv\; 2\pi p_{\perp} dp_{\perp}\, dp_z \;\to\; 2\pi\,\frac{eB\hbar}{c}\, dp_z \sum_n \qquad (43)$$

defines the phase space element for charges in QED in external $\mathbf{B}$ fields.

3 Relativistic Collisions and Decay

In space physics, astrophysics and particle physics, collision integrals are frequently employed and a core element controlling their evaluation concerns the relativistic kinematics of various interactions. We will consider two common examples: particle decay and two-body collisions.

3.1 Decay of Particles

Consider a spontaneous decay of a species into two particles, $M \to 1 + 2$. The masses of these will be denoted M, m_1 and m_2. Energy conservation dictates that $M \geq m_1 + m_2$. Conservation of p^{μ} in this decay constrains both the energy and momentum. The simplest case is the decay of a particle at rest in the **centre of momentum** (CM) frame (in units of $c = 1$): **L&L Sec. 11**

$$M \;=\; \mathcal{E}_1 + \mathcal{E}_2 \quad , \quad \mathbf{p}_1 + \mathbf{p}_2 \;=\; 0 \quad . \qquad (44)$$

As the 3D momenta are equal and opposite, one has

$$\mathcal{E}_1^2 - m_1^2 \;=\; \mathcal{E}_2^2 - m_2^2 \quad\Rightarrow\quad \mathcal{E}_1^2 - \mathcal{E}_2^2 \;=\; m_1^2 - m_2^2 \quad . \tag{45}$$

These can be combined with the mass/energy identity to give

$$\mathcal{E}_1 \;=\; \frac{M^2 + m_1^2 - m_2^2}{2M} \quad , \quad \mathcal{E}_2 \;=\; \frac{M^2 + m_2^2 - m_1^2}{2M} \quad , \tag{46}$$

from which the magnitude $p = |\mathbf{p}_1| = |\mathbf{p}_2|$ of the vector momenta can be deduced:

$$4M^2\, p^2 \;=\; \Bigl(M^2 - (m_1 + m_2)^2 \Bigr) \Bigl(M^2 - (m_1 - m_2)^2 \Bigr) \quad . \tag{47}$$

This is clearly a Lorentz invariant, so that it can be used to determine the momenta and energies of the decay products for moving masses M .

$\boxed{\textbf{Plot:}}$ Draw diagram of free decay into two species

• An example of a reaction to which this applies is $\pi^0 \to \gamma\gamma$, so that the photon momenta are $p = E/c = m_\pi/2$, i.e. around 67 GeV/c. Discuss SNR and cosmic ray context in the light of *Fermi*-LAT discoveries.

If the decaying mass is actually in flight, the energy and momenta are most simply determined by a Lorentz transformation from the mass's rest frame Then, the vector direction of the decay product momenta samples an isotropic distribution (unless there is an external **E** or **B** field). Once this direction is determined probabilistically, the boost can be applied to specify all quantities in the laboratory (L) frame.

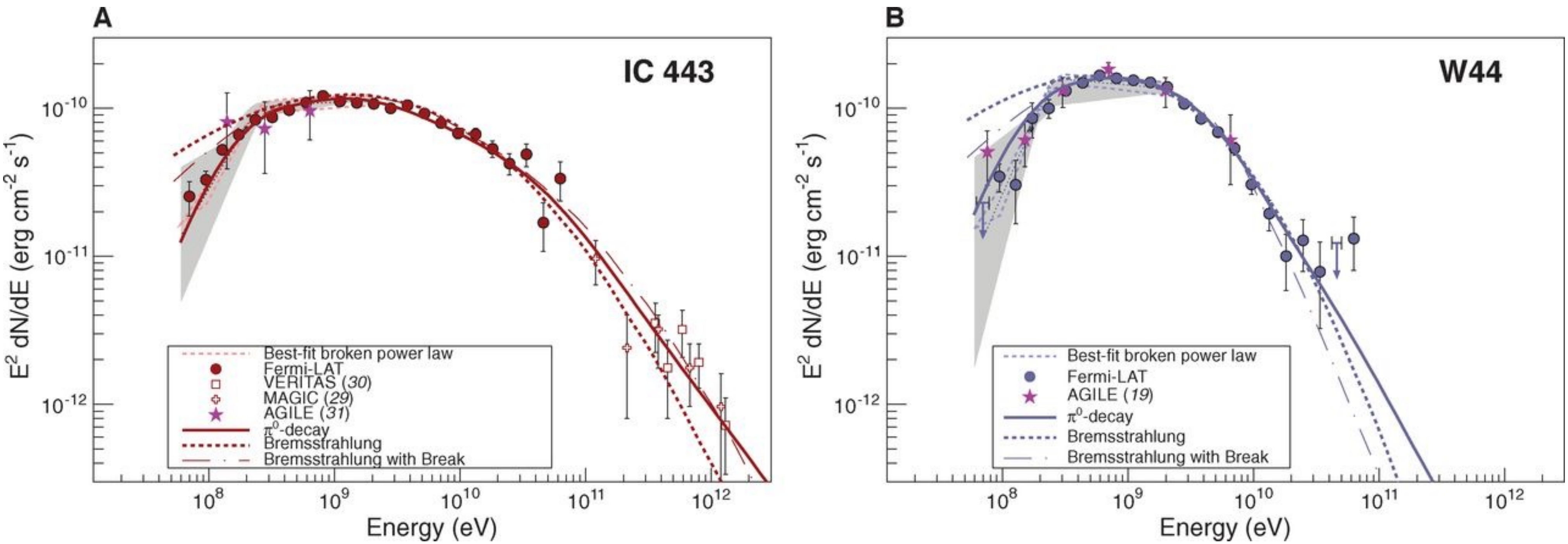
$*$ The distribution of rest frame momentum angles defines a flat-topped distribution of the energies of the decay products in the L-frame.

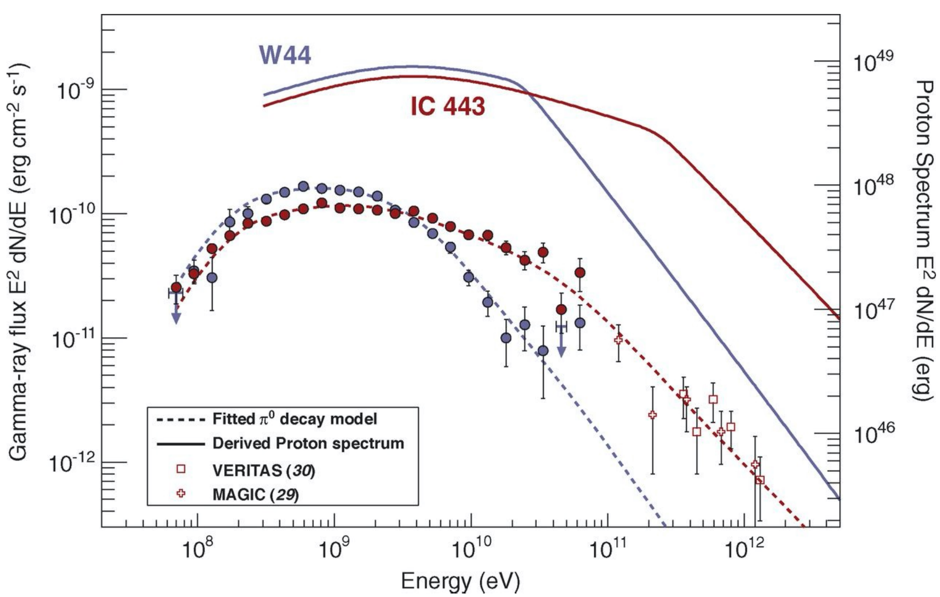
$\boxed{\textbf{Plot:}}$ π^0 Gamma Rays from Supernova Remnants

• A second example is $\pi^\pm \to \mu^\pm \nu_e$, as is prolific in the CMS detector at the LHC. Talk about IceCube and detection of atmospheric neutrinos as a background signal to VHE cosmic ν observation.

$*$ Also discuss $n \to p + e^- + \overline{\nu}_e$, for which measurement precision is sufficient to prove the existence of the neutrino through <u>missing momentum</u>, but insufficient to measure its small mass, only bound it.

π^0 Gamma-Rays from Supernova Remnants





- *Top:* Ackermann et al. (Science 2013): *Fermi*-LAT sees down to 60 MeV with 4 years of data.
- Bump seen at same energies near 67 MeV in 2 remnants.
- Inferred cosmic ray proton distribution is curved (*left*).

3.2 Invariant Cross Section

Collisions of particles are characterized by their **invariant cross sections** σ, which determine the number of collisions between beams of interacting particles. The collision rate can be cast in the familiar form **L&L Sec. 12**

$$dN \;=\; \sigma\, v_{\rm rel} n_1 n_2\, dV\, dt \quad , \quad \text{for} \quad dV\, dt \;=\; d^4x \quad . \tag{48}$$

Here n_i are the number densities of the colliding species, and $v_{\rm rel}$ is their relative speed. This rate is *nominally written in the rest frame of one of the species*, say particle 2, and thus <u>is not yet covariant</u>.

We seek to extend this to a fully covariant form, noting that neither n_1 nor n_2 is a relativistic invariant. However, dN is a true scalar, as the total number of particles cannot vary from frame to frame. Thus we posit

$$dN \;=\; \mathcal{A}\, n_1 n_2\, dV\, dt \quad , \tag{49}$$

with the goal of determining the coefficient $\mathcal{A}$ so as to render this expression covariant, with $\mathcal{A} \to \sigma v_{\rm rel}$ in the rest frame of one of the species. Clearly $\mathcal{A} n_1 n_2$ must be a Lorentz invariant.

If n_0 and dV_0 are the number density and volume element of one species in its rest frame, then the number $n\, dV = n_0\, dV_0$ is a Lorentz invariant. Thus

$$n \;=\; \gamma n_0 \tag{50}$$

defines the **density compression** in a Lorentz boost.

- We already know that $\mathcal{E} \;=\; \gamma \mathcal{E}_0 \;=\; \gamma m c^2$ gives the energy relationship to the rest frame, so it follows that $\mathcal{A}\, \mathcal{E}_1\, \mathcal{E}_2$ must also be a Lorentz invariant. This is a convenient equivalence because we can now work with four-momentum variables. Scaling this by the four-momentum scalar product $p_{1\mu} p_2^{\mu}$, we arrive at an invariant (<u>in units of $c = 1$</u>)

$$\mathcal{A}\, \frac{\mathcal{E}_1 \mathcal{E}_2}{p_{1\mu} p_2^{\mu}} \;=\; \mathcal{A}\, \frac{\mathcal{E}_1 \mathcal{E}_2}{\mathcal{E}_1 \mathcal{E}_2 - \mathbf{p}_1 \cdot \mathbf{p}_2} \;=\; \text{const.} \;=\; \sigma v_{\rm rel} \tag{51}$$

The evaluation of the constant is achieved in the rest frame of either particle, and so is just σ times the relative speed. Therefore we have found $\mathcal{A}$.

To give the collision rate expression its final form, we need to express the relative velocity $v_{\rm rel} = \beta_{\rm rel} c$ in both covariant and vector form. In the rest frame of either particle, since $\gamma = (1-\beta_{\rm rel}^2)^{-1/2}$ is the Lorentz factor of the moving particle, as $\mathbf{p}_1 \cdot \mathbf{p}_2 = 0$,

$$p_{1\mu} p_2^\mu \;=\; \frac{m_1 m_2}{\sqrt{1-\beta_{\rm rel}^2}} \quad \Rightarrow \quad \beta_{\rm rel} \;=\; \sqrt{1 - \left(\frac{m_1 m_2}{p_{1\mu} p_2^\mu}\right)^2} \quad . \tag{52}$$

This is the covariant form. To realize a vector form, for general frames,

$$p_{1\mu} p_2^\mu \;=\; \mathcal{E}_1 \mathcal{E}_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 \;=\; \gamma_1 \gamma_2 m_1 m_2 (1 - \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2) \quad , \tag{53}$$

and this can be used to re-express the relative speed:

$$\beta_{\rm rel} \;=\; \sqrt{1 - \frac{(1-\beta_1^2)\,(1-\beta_2^2)}{(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)^2}} \;\equiv\; \frac{\sqrt{(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2)^2 - (\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2)^2}}{1 - \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2} \quad . \tag{54}$$

The second form is obtained by using standard vector identities.

Assembling all the various pieces, the reaction rate can be written in covariant form thus:

$$dN \;=\; \sigma\, c\, \frac{\sqrt{(p_{1\mu} p_2^\mu)^2 - m_1^2 m_2^2}}{\mathcal{E}_1 \mathcal{E}_2}\, n_1 n_2\, dV\, dt \quad , \tag{55}$$

with $n_1 n_2 / (\mathcal{E}_1 \mathcal{E}_2)$ being a Lorentz invariant, or equivalently in vector form

$$dN \;=\; \sigma\, c\, \sqrt{(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2)^2 - (\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2)^2}\; n_1 n_2\, dV\, dt \quad . \tag{56}$$

This form was derived by Wolfgang Pauli in 1933. If the two velocities are collinear, as they are in CM frame collisions, then $\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2 = \mathbf{0}$, and the rate is proportional to a familiar Galilean relative velocity factor $|\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2|\, c$.