Consider the simpler case of one set of canonical variables, q and $\dot{q}$. Let $q(t)$ be the variational solution for the actual trajectory. Consider perturbations about this equilibrium solution, $q \to q + \delta q$ and $\dot{q} \to \dot{q} + \delta\dot{q}$. Fix the endpoint values at points a and b, so that no variations arise due to them, thereby fixing $\delta q(t_a) = 0 = \delta q(t_b)$. The variation in the action becomes

$$\delta S \;=\; \int_{t_a}^{t_b} L(q + \delta q,\, \dot{q} + \delta\dot{q},\, t)\, dt - \int_{t_a}^{t_b} L(q,\, \dot{q},\, t)\, dt \quad . \tag{2}$$

A Taylor series expansion of the integrand in the first integral then yields

$$\delta S \;=\; \int_{t_a}^{t_b} \left(\frac{\partial L}{\partial q}\, \delta q + \frac{\partial L}{\partial \dot{q}}\, \delta\dot{q} \right) dt \quad . \tag{3}$$

Since $\delta\dot{q} = d(\delta q)/dt$, we can integrate the second term by parts:

$$\int_{t_a}^{t_b} \frac{\partial L}{\partial \dot{q}} \left(\frac{d}{dt}\, \delta q \right) dt \;=\; \left[\frac{\partial L}{\partial \dot{q}}\, \delta q \right]_{t_a}^{t_b} - \int_{t_a}^{t_b} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q\, dt \tag{4}$$

The first term on the RHS is identically zero since the endpoints are fixed. The variation of the action then takes the form

$$\delta S \;=\; -\int_{t_a}^{t_b} \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} \right\} \delta q\, dt \quad , \tag{5}$$

which can only be zero if the factor in curly braces is zero. Accordingly,

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) \;=\; \frac{\partial L}{\partial \mathbf{q}} \quad ,} \tag{6}$$

which define the **Euler-Lagrange equations** (E-L) of motion; this result has been routinely extended to the more general vector form.

- If L has the dimensions of energy and q the dimensions of length, then each term in the E-L result has the *dimensions of force*. Accordingly, define

$$\mathbf{p} \;\equiv\; \frac{\partial L}{\partial \dot{\mathbf{q}}} \tag{7}$$

to be the **canonical momentum**, conjugate to the space variable $\mathbf{q}$. We can explicitly solve this for $\dot{\mathbf{q}}(\mathbf{p})$.

• This should be familiar from quantum theory. Equations of motion for a system intimately couple distance and momentum, which in a Fourier representation leads to the appearance of complex exponentials of the form $\exp\{i(\mathbf{p}\cdot\mathbf{x})/\hbar\}$. Eventually, the extension of this to four-vector formalism automatically implies additional time-dependent factors $\exp\{-iEt/\hbar\}$.

Then, to extend this slightly further, define a **Hamiltonian** via

$$H(\mathbf{q},\, \mathbf{p},\, t) \;=\; \mathbf{p}\cdot\dot{\mathbf{q}} - L(\mathbf{q},\, \dot{\mathbf{q}},\, t) \quad . \tag{8}$$

In ensembles of charged particles and electromagnetic fields, in this course, the Hamiltonian will represent the total energy $\mathcal{E}$ of the system, kinetic plus potential. Using it, Eqs. (6) and (7) can be recast as **Hamilton's equations**:

$$\boxed{\dot{\mathbf{q}} \;=\; \frac{\partial H}{\partial \mathbf{p}} \quad , \quad \dot{\mathbf{p}} \;=\; -\frac{\partial H}{\partial \mathbf{q}} \quad .} \tag{9}$$

These are conjugate equations for conjugate variables.

• If the Lagrangian is explicitly independent of time, i.e. $\partial L/\partial t = 0$, then so also is H ($\partial H/\partial t = 0$), and the total energy is conserved: $d\mathcal{E}/dt = 0$:

$$\frac{d\mathcal{E}}{dt} \;=\; \dot{\mathbf{p}}\cdot\frac{\partial H}{\partial \mathbf{p}} + \dot{\mathbf{q}}\cdot\frac{\partial H}{\partial \mathbf{q}} + \cancelto{0}{\frac{\partial H}{\partial t}} \;=\; \dot{\mathbf{p}}\cdot\dot{\mathbf{q}} - \dot{\mathbf{q}}\cdot\dot{\mathbf{p}} \;=\; 0 \quad , \tag{10}$$

using Hamilton's equations to substitute.

• The Hamiltonian forms the basis for the **virial theorem** for bound systems of particles in central force potentials. Examples of such ensembles are free electrons in a metallic lattice, and self-gravitating clusters of stars.

∗ $\langle H + L\rangle$ represents the time rate of change of the moment of inertia of an ensemble of particles. Accordingly, it connects to the total angular momentum of the system.

• A covariant formulation for the E-L equations is easy to develop from this exposition, but will not be expounded here.

1.1 Relativistic Action for Free Particles

To extend the principle of least action to relativistic systems, we need the action to be defined so that it is a Lorentz invariant. For a free particle subject to no forces, the path must intrinsically not depend on the inertial frame, yet the path's description can be frame-dependent. The only path variable that is a scalar is the line element ds, so that the logical and simplest choice for the action (free particle only) is **L&L Sec. 8**

$$S \;=\; -\alpha \int_a^b ds \;=\; \int_{t_a}^{t_b} L\, dt \quad , \tag{11}$$

where α is a constant to be determined. This is well-constructed, since we know that the integral has its maximum value along a straight world line. Choosing $\alpha > 0$ gives S at a minimum along the **geodesic**. If the particle moves with speed v, then $ds = cdt/\gamma$, and the action is

$$S \;=\; -\alpha \int_a^b c\sqrt{1 - \frac{v^2}{c^2}}\, dt \quad \Rightarrow \quad L \;=\; -\alpha c\sqrt{1 - \frac{v^2}{c^2}} \quad . \tag{12}$$

The inference of the form for the Lagrangian follows from a comparison with the (by now) familiar definition in Eq. (1). Taylor expansion in the non-relativistic $v \ll c$ limit yields

$$L \;\approx\; -\alpha c + \frac{\alpha v^2}{2c} \quad . \tag{13}$$

The constant term is immaterial to the Euler-Lagrange formalism (because its derivatives are zero), and so to render this compatible with the classical kinetic energy form, $L = mv^2/2$, we have $\alpha = mc$. The action and Lagrangian *for a free particle* are then

$$S \;=\; -mc \int_a^b ds \quad \text{and} \quad L \;=\; -mc^2\sqrt{1 - \frac{v^2}{c^2}} \quad . \tag{14}$$

The Lagrangian $L = -mc^2/\gamma$ therefore *possesses a rest mass energy contribution*, a feature particular to special relativity.

- More complicated covariant forms for actions could be chosen, but these would not match experimental properties.

2 Relativistic Momentum and Energy

Now that the relativistic Lagrangian for a free particle has been formulated, we can write down the vector momentum with the aid of Eq. (7):

L&L Sec. 9

$$\mathbf{p} \;\equiv\; \frac{\partial L}{\partial \mathbf{v}} \;=\; \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \;\equiv\; \gamma m \mathbf{v} \quad . \tag{15}$$

Here we have written $v^2 = \mathbf{v} \cdot \mathbf{v}$ in the expression for L. This reduces to our familiar form for non-relativistic speeds $v \ll c$, and diverges to infinity as $v \to c$. $\Rightarrow$ *Massive particles cannot reach the speed of light*, no matter how large the acceleration is. The force can then be simply written down:

$$\frac{d\mathbf{p}}{dt} \;=\; \gamma m \frac{d\mathbf{v}}{dt} + \frac{d\gamma}{dt} m\mathbf{v} \quad . \tag{16}$$

The *first term is germane to circular motion*, where forces do no work, and only the direction of the velocity vector changes, not its magnitude. The *second term pertains to purely linear acceleration*, where (in conjunction with the first term) relativistic charges at $|v| \sim c$ can continue to gain momentum *ad infinitum*: this is the domain of particle accelerators!

The relativistic energy $\mathcal{E}$ can be defined directly using the Hamiltonian:

$$\mathcal{E} \;=\; H \;\equiv\; \mathbf{p} \cdot \mathbf{v} - L \;=\; \gamma m v^2 + \frac{mc^2}{\gamma} \;=\; \gamma m c^2 \quad . \tag{17}$$

Clearly, as $v \to 0$, $\mathcal{E} \to mc^2$ and we term this the **rest mass energy** of the particle. The first order correction to this in the $v \ll c$ limit is the familiar non-relativistic kinetic energy $K = mv^2/2$. As $v \to c$, $\mathcal{E} \to \infty$ and again, on energetic grounds, *massive particles cannot reach the speed of light.*

The relationship between energy and momentum necessarily requires squares because of their dependence on speed. One quickly arrives at

$$\mathcal{E}^2 - p^2c^2 \;=\; \left(mc^2\right)^2 \quad \Rightarrow \quad \mathcal{E} \;=\; \sqrt{(pc)^2 + (mc^2)^2} \quad , \tag{18}$$

using $\gamma^2 - \gamma^2\beta^2 = 1$. The $p \ll mc$ limit yields the familiar non-relativistic kinetic energy form $K = p^2/2m$ as the correction to the rest mass energy.

• It must be emphasized that these relations apply not only to microscopic elementary particles, but also to macroscopic objects that possess both the "raw" mass of constituents, but also the binding energy associated with attractive forces of molecules and lattices. This means that these forces and binding energies (e.g. electrostatics) will be subject to the same laws of special relativity — the formalism of four-vectors will accomplish this.

• This energy-momentum relation is easily adapted to treat massless light, for which $\mathcal{E} = pc$. To formulate the Lagrangian and Hamiltonian for massless particles, we have to divide the preceding developments through by mass, so that the Hamiltonian describes energy per unit mass.

It is now the goal to form a four-vector combining both momentum and energy. Given their relationship in Eq. (18), it is obvious that they should be intimately connected, and in fact from this one could deduce the appropriate 4-vector. We can start with the principle of least action,

$$S \;=\; -mc\int_a^b ds \;=\; -mc\int_a^b u_\mu\, dx^\mu \;\;\Rightarrow\;\; \delta S \;=\; -mc\int_a^b u_\mu\, \delta(dx^\mu) \quad . \tag{19}$$

Here we have used

$$ds^2 \;=\; dx_\mu dx^\mu \quad \Rightarrow \quad ds \;\to\; \frac{dx_\mu}{ds}\, dx^\mu \;=\; u_\mu\, dx^\mu \quad . \tag{20}$$

Using $\delta(dx^\mu) = d(\delta x^\mu)$ and integrating by parts, we obtain

$$\delta S \;=\; -mc\Big[u_\mu\, \delta x^\mu\Big]_a^b + mc\int_a^b \delta x_\mu\, \frac{du_\mu}{ds}\, ds \quad . \tag{21}$$

The variational principle for fixed endpoints automatically yields zero for the first term outside the integral. Accordingly, the equations of motion for a free particle are $du_\mu/ds = 0$, as before. If we adiabatically move over to cases of slightly different velocity,[1] then $(\delta x^\mu)_a = 0$ as before, but $(\delta x^\mu)_b \neq 0$. The "perturbation" to the action over varied paths satisfies

$$\delta S \;=\; -mc\, u_\mu\, \delta x^\mu \quad , \quad \text{suggesting} \quad p_\mu \;\equiv\; -\frac{\partial S}{\partial x^\mu} \;=\; mc\, u_\mu \quad . \tag{22}$$

The second part of the second identity here comes from the derivative of the non-perturbative form of Eq. (19), i.e. the indefinite form, assuming that the

[1]Draw two straight wordlines with different $\mathbf{p}$ incurring δx^μ.

spacetime locale b is freely varied. This relation serves as the definition of the **four-momentum** p^μ, a 4-vector that places energy and momentum on an equal footing in relativistic spacetime. The contravariant and covariant components are

$$p^\mu \;=\; \left(\frac{\mathcal{E}}{c}\,,\,\mathbf{p}\right) \quad , \quad p_\mu \;=\; \left(\frac{\mathcal{E}}{c}\,,\,-\mathbf{p}\right) \quad , \tag{23}$$

and it follows that the 4-momentum possesses a magnitude defined by

$$p_\mu p^\mu \;=\; (mc)^2 u_\mu u^\mu \;=\; (mc)^2 \quad , \tag{24}$$

clearly a Lorentz invariant. This identity also confirms Eq. (18). Since p^μ is a four-vector, the Lorentz transformation equations for momentum and energy are simply established (and experimentally verified):

$$p_x \;=\; \gamma\left(p'_x + \beta\,\frac{\mathcal{E}'}{c}\right) \,, \quad p_y \;=\; p'_y \,, \quad p_z \;=\; p'_z \,, \quad \mathcal{E} \;=\; \gamma\Bigl(\mathcal{E}' + \beta p'_x\, c\Bigr) \,. \tag{25}$$

Finally, the four-vector for force can be defined using derivatives of the four-momentum. This gives a **four-force**

$$f^\mu \;=\; \frac{dp^\mu}{ds} \;=\; mc\,\frac{du^\mu}{ds} \;\equiv\; mc\,a^\mu \quad . \tag{26}$$

This is a logical choice as it is proportional to the four-acceleration, and therefore also satisfies the orthogonality property $f_\mu u^\mu = 0$. The four-vector expanded form then becomes

$$f^\mu \;=\; \frac{\gamma}{c}\left(\frac{\mathbf{f}\cdot\mathbf{v}}{c}\,,\,\mathbf{f}\right) \quad , \tag{27}$$

where $\mathbf{f} = d\mathbf{p}/dt$ is the traditional three-vector force. The time component is clearly proportional to rate of the work done, and all components possess time-dilation factors courtesy of $ds = cdt/\gamma$.

2.1 Angular Momentum

To complete the pedagogy on dynamical quantities governed by conservation laws, it is natural to consider angular momentum $\mathbf{J} = \mathbf{r} \times \mathbf{p}$. Since $\mathbf{r}$ and $\mathbf{p}$ are both vector quantities, one intuitively expects that angular momentum $\mathbf{J}$ must be described by a tensor. The conservation of $\mathbf{J}$ is a consequence of the invariance of a system under rotations; thus we explore this rotational symmetry in the extended dimensions of spacetime.

L&L Sec. 14

Let $\delta\Omega_{\mu\nu}$ be an infinitesimal 4D rotation that is thus a Lorentz transformation. Coordinates will change according to

$$(x^\mu)' - x^\mu \;\equiv\; \delta x^\mu \;=\; x_\nu\,\delta\Omega^{\mu\nu} \quad . \tag{28}$$

Since x^μ is a four-vector, the scalar product $x_\mu x^\mu$ is a Lorentz invariant under such rotations. Expressing this using Eq. (28), and retaining only the leading order (linear) terms in $\delta\Omega_{\mu\nu}$, one arrives at a conservation relation:

$$x'_\mu\,(x^\mu)' \;=\; x_\mu x^\mu \quad\Rightarrow\quad x^\mu x^\nu\,\delta\Omega_{\mu\nu} \;=\; 0 \quad . \tag{29}$$

This is a contraction of two tensors, one of them being $x^\mu x^\nu$, a symmetric tensor. It follows that $\delta\Omega_{\mu\nu}$ must be anti-symmetric, i.e. $\delta\Omega_{\nu\mu} = -\delta\Omega_{\mu\nu}$.

Now form the variation of the action δS due to this coordinate rotation for a world-line path of a particle between points a and b:

$$\delta S \;=\; -\big[p^\mu\,\delta x^\nu\big]_a^b \;=\; -\delta\Omega_{\mu\nu}\,\big[p^\mu x^\nu\big]_a^b \;=\; \delta\Omega_{\mu\nu}\,\big[p^\nu x^\mu\big]_a^b \tag{30}$$

using the definition of four-momentum. The last form exploits the anti-symmetry of the infinitesimal rotation tensor using a re-labelling $\mu \leftrightarrow \nu$. Averaging the two forms yields

$$\delta S \;=\; \frac{1}{2}\,\delta\Omega_{\mu\nu}\,\big[J^{\mu\nu}\big]_a^b \quad , \quad J^{\mu\nu} \;=\; \sum_i \big(x^\mu p^\nu - x^\nu p^\mu\big) \tag{31}$$

Here we have introduced a sum over all particles. Since the action is a Lorentz invariant under such coordinate rotations, $\delta S = 0$ with a and b fixed, so that the coefficient of $\delta\Omega_{\mu\nu}$ must be zero. From this we derive a conservation law, and interpret $J^{\mu\nu}$ as the **4-tensor for angular momentum**.

- The space-space components are clearly in an $\mathbf{r}\times\mathbf{p}$ form, and constitute the axial vector $\mathbf{J}$ for the ordinary angular momentum. The time-space components are proportional to the evolving displacement, $\mathbf{v}t-\mathbf{r}$, a polar vector, and so describe the center of inertia of the ensemble. Using $p_t = \mathcal{E}/c$,

$$J^{\mu\nu} \;=\; \begin{pmatrix} 0 & p_t(v_x t - x) & p_t(v_y t - y) & p_t(v_z t - z) \\ p_t(x - v_x t) & 0 & J_z & -J_y \\ p_t(y - v_y t) & -J_z & 0 & J_x \\ p_t(z - v_z t) & J_y & -J_x & 0 \end{pmatrix} \quad . \tag{32}$$