- In every tensor equation (e.g. Maxwell's equations – to come), all terms must contain identical and identically-placed (i.e., raised or lowered) free indices, as opposed to *dummy indices* that are summed over. Free indices can be shifted up or down to derive alternative forms of the tensor equation, but such operations must be executed simultaneously to all terms.

The property that tensors map like products of 4-vectors under Lorentz transformations immediately identifies the transformation relations for a 4-tensor in boosting between the K and K' frames:

$$T_{\mu\nu} \;=\; \frac{\partial x_\mu}{\partial {x'}_\alpha} T'_{\alpha\beta} \frac{\partial x_\nu}{\partial {x'}_\beta} \;\equiv\; \mathbf{\Lambda}_\mu^\alpha \, T'_{\alpha\beta} \, \mathbf{\Lambda}_\nu^\beta \quad , \tag{59}$$

where the primes denote evaluation of quantities in the K' frame.

$*$ Observe that the combined pre- and post-multiplication protocol by Lorentz boost matrices is needed in order to preserve proper manipulations of tensor operations on vectors. This is immediately deducible by constructing the tensors as products of column and row vectors.

$*$ Observe also that the boost transformations are also rank 2 tensors, and the placement of the indices, covariant in both Jacobian denominators, yields contravariant α and β indices in the corresponding $\mathbf{\Lambda}$. This preserves the signs of the space elements to the $\mathbf{\Lambda}_\mu^\alpha$ that are present in Eq. (45).

- One can form scalars from tensors, just as we did for 4-vectors, by summing over the indices. This reduces the number of free indices in an operation called **index contraction**, an example of which is

$$T^\mu_{\ \mu} \;=\; \sum_{\mu=0}^{3} T^\mu_{\ \mu} \quad , \tag{60}$$

which forms the **trace** of the tensor matrix. In general, contracting any pair of indices reduces the rank of a tensor by 2.

5.1 Special Tensors

Two special tensors are now introduced. The first is the **unit 4-tensor** δ^μ_ν,

$$\delta^\mu_\nu \;=\; \begin{cases} 1 \quad , \text{ if } \mu \;=\; \nu \quad , \\ 0 \quad , \text{ if } \mu \;\neq\; \nu \quad . \end{cases} \tag{61}$$

This evinces the property that for any four-vector A^ν,

$$\delta^\mu_\nu A^\nu \;=\; A^\mu \quad . \tag{62}$$

One can then raise or lower the indices of this tensor to obtain the **metric tensor** ($g^{\mu\nu}$ in contravariant form, $g_{\mu\nu}$ in covariant form). In flat spacetime, the metric tensor assumes the familiar **Minkowski metric** form

$$g^{\mu\nu} \;=\; g_{\mu\nu} \;\equiv\; \eta^{\mu\nu} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \;=\; \eta_{\mu\nu} \quad . \tag{63}$$

This now serves as the index raising and lowering operator for 4-vectors:

$$\begin{aligned} g_{\alpha\beta} A^\beta \;&=\; A_\alpha \quad , \quad g^{\alpha\beta} A_\beta \;=\; A^\alpha \\ A^\nu A_\nu \;=\; g_{\alpha\beta} A^\alpha A^\beta \;&=\; g^{\alpha\beta} A_\alpha A_\beta \quad . \end{aligned} \tag{64}$$

- These tensors are invariant under Lorentz transformations:

$$g_{\mu\nu} \;=\; \mathbf{\Lambda}^\alpha_\mu \, g_{\alpha\beta} \, \mathbf{\Lambda}^\beta_\nu \quad . \tag{65}$$

This special property is easily confirmed using Eq. (45) for the flat spacetime case $g_{\mu\nu} \to \eta_{\mu\nu}$, and is a result that applies in all coordinate systems, for example Cartesian and spherical polar configurations.

- This invariance suggests that metric tensors are tied to the spacetime geometry. Consider now the space four-vector $A^\nu \to dx^\nu$. Then we have

$$ds^2 \;=\; dx^\nu dx_\nu \;=\; g_{\alpha\beta} dx^\alpha dx^\beta \quad , \tag{66}$$

which is the differential form of Eq. (36) for flat spacetime. In a gravitational field, $g_{\mu\nu}$ no longer assumes this simple form, and can in fact be non-diagonal, for example capturing the frame-dragging arising in the rotating Kerr metric.

• Another useful special tensor is the **completely antisymmetric unit tensor of 4th rank**, $\epsilon^{\alpha\beta\gamma\delta}$. This is the tensor whose components change sign under the interchange of any two indices. This antisymmetry constraint render all components with two identical indices necessarily zero. The normalization property sets its nonzero components to be ± 1.

∗ Yet we still have a freedom of overall sign, so we set $\epsilon^{0123} = 1$.

∗ There are a total of 24 non-vanishing components out of the 256.

[*Reading Assignment: Pages 17-18 on the antisymmetric unit tensor*]

5.2 Anti-symmetric Tensors

• An anti-symmetric tensor has 12 nonzero components, and the antisymmetry constraint yields only six independent elements. Therefore, we can write it in the form

$$T^{\mu\nu} = \begin{pmatrix} 0 & p_x & p_y & p_z \\ -p_x & 0 & -a_z & a_y \\ -p_y & a_z & 0 & -a_x \\ -p_z & -a_y & a_x & 0 \end{pmatrix} \quad , \tag{67}$$

without loss of generality. This implies that the tensor can be considered as being constructed *using two purely spatial vectors*, $\mathbf{p}$ and $\mathbf{a}$. We represent this as two sets, one for each of the covariant and contravariant forms:

$$T^{\mu\nu} = (\mathbf{p}, \mathbf{a}) \quad , \quad T_{\mu\nu} = (-\mathbf{p}, \mathbf{a}) \quad . \tag{68}$$

If we perform a reflection of all the spatial coordinates, then the components with a single time index (0) switch sign, but those with two spatial indices (i.e. no temporal ones) do not. This is tantamount to the vector $\mathbf{p}$ *reversing directions* under reflections (*odd parity*); it is then called a **polar vector** (hence the symbol).

∗ Examples of polar vectors in physics include linear momentum $\mathbf{p}$ and the electric field $\mathbf{E}$ (e.g. in a dipole configuration).

On the other hand, the **a** vector does not change signs under such a reflection operation (*even parity*). If it were a linear superposition of two polar vectors, it would then itself be polar. Instead, what if it is the cross product of two vectors: $\mathbf{a} = \mathbf{b} \times \mathbf{c}$? If both **b** and **c** are both polar, then **a** does not change sign under this inversion, reproducing its even parity. It is then termed an **axial vector** (hence the symbol).

Plot: Polar and axial vectors under 1D reflections

∗ Angular velocity vectors $\boldsymbol{\omega} \times \mathbf{r}$ describing rotations, orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ are examples of axial vectors.[3]

The cross product serves as a convenient representation of axial vectors, since then the purely spatial elements of the tensor assume a form $a_k = b_i c_j - b_j c_i$ for cyclic permutation of the indices $i, j, k \rightarrow x, y, z$. This then re-distributes the spatial indices in the positions that exactly match the a_k elements given in $T^{\mu\nu}$ above. We emphasize that this is not a unique choice, but *it is a convenient one for electromagnetism*, for which cross product structure in the form of curls appears in the field equations.

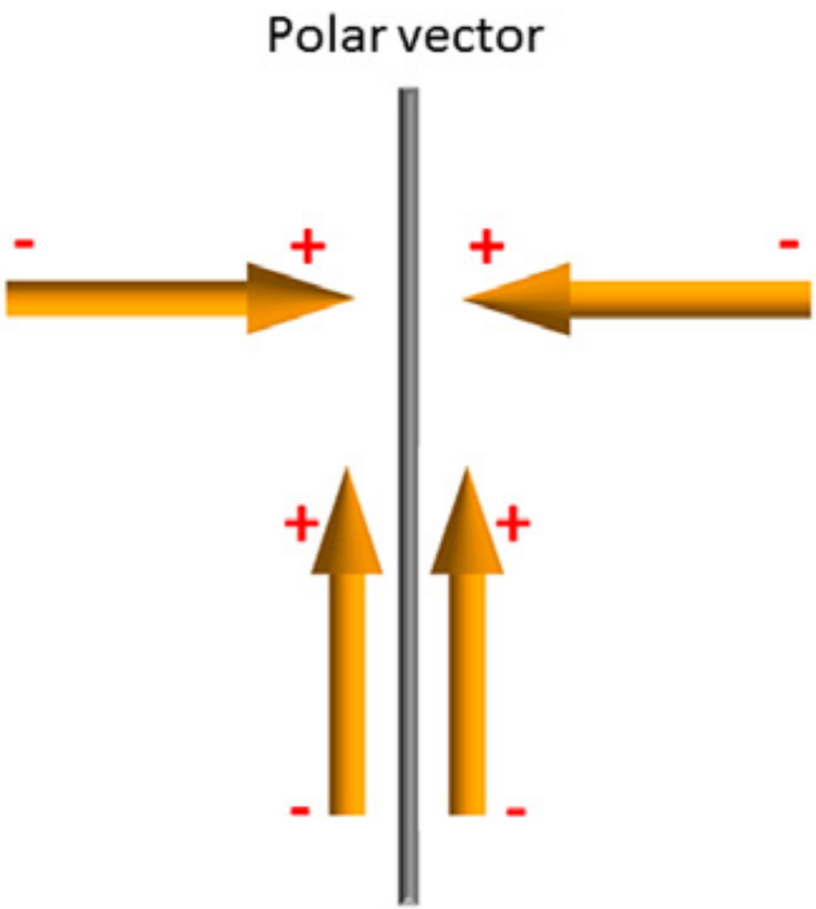
∗ Thus, we expect that the fundamental tensor for electromagnetism will be anti-symmetric, with the electric field components occupying the **p** elements and the magnetic field components occupying the **a** elements.

- Any tensor $T_{\mu\nu}$ can be decomposed into the sum of its symmetric, $(T_{\mu\nu} + T_{\nu\mu})/2$, and anti-symmetric, $(T_{\mu\nu} - T_{\nu\mu})/2$ parts. These parts can often constitute different physical content within one umbrella description, an example being the matter and electromagnetic components of the relativistic **energy-momentum tensor**.

[3] See the Feynman Lectures at `http://www.feynmanlectures.caltech.edu/I_52.html`

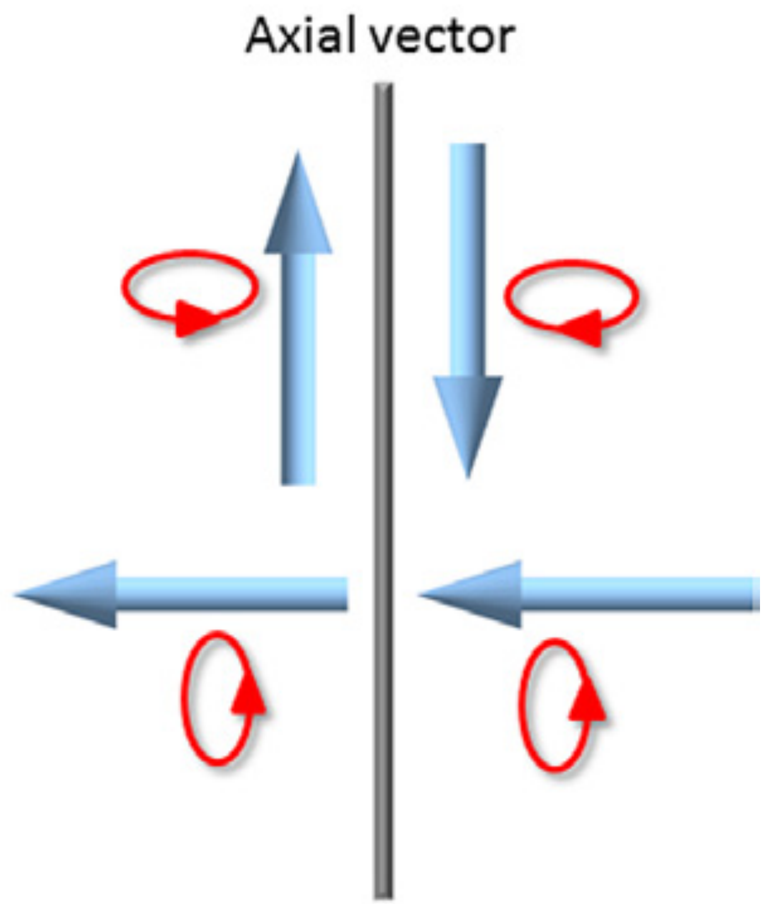
Polar and Axial Vectors

Credit: Damay 2015 *J. Phys. D: Appl. Phys.* **48** 504005



mirror

Electric dipole



mirror

Magnetic dipole

5.3 Differentiation and Integration

It is straightforward to generalize the gradient operation to posit a **four-gradient** of a scalar function ϕ:

$$\begin{aligned} \partial_\mu \phi \;&\equiv\; \frac{\partial \phi}{\partial x^\mu} \;=\; \left(\frac{1}{c}\frac{\partial \phi}{\partial t}, \nabla \phi \right) \quad , \\ \partial^\mu \phi \;&\equiv\; \frac{\partial \phi}{\partial x_\mu} \;=\; \left(\frac{1}{c}\frac{\partial \phi}{\partial t}, -\nabla \phi \right) \quad . \end{aligned} \tag{69}$$

These are respectively in covariant and contravariant forms, and define true 4-vectors. Observe the shorthand notation of the differentiation. Lorentz transformation of such forms will dictate their use as a compact tool for representing key results in electrodynamics.

- We can also differentiate four-vectors, and provided that we maintain familiar protocols with the indices, the result is either a scalar or a tensor. We usually differentiate a contravariant 4-vector using a covariant derivative, and this is the **four-divergence**:

$$\partial_\mu A^\mu \;\equiv\; \frac{\partial A^\mu}{\partial x^\mu} \quad . \tag{70}$$

Such forms will eventually appear in our covariant construction of Maxwell's and other electrodynamic equations.

L&L talk about four types of integrations that can be constructed using spacetime dimensions. These increase in dimensions. The simplest is just an integral over a curve in four-space, so that the element of integration is just the line element, i.e. dx_α. For the others, L&L appeal to our familiarity with 3D spatial integrations, and analogies extend to spacetime. In general, the integration elements are **Jacobians** for the coordinate transformation from the curvilinear surface involved to the projected plane in the restricted dimensions. For example, in 2D, one integrates using a surface element

L&L p. 21

$$dS_{\alpha\beta} \;\equiv\; \begin{vmatrix} dx_\alpha & dx'_\alpha \\ dx_\beta & dx'_\beta \end{vmatrix} \;=\; dx_\alpha dx'_\beta - dx'_\alpha dx_\beta \quad , \tag{71}$$

employing the familiar determinant notation for the Jacobian. N.B. *primes here denote alternative coordinate basis* as opposed to another inertial frame.

In 3D, the surface element projection is also a determinant:

$$dS_{\alpha\beta\gamma} \equiv \begin{vmatrix} dx_\alpha & dx'_\alpha & dx''_\alpha \\ dx_\beta & dx'_\beta & dx''_\beta \\ dx_\gamma & dx'_\gamma & dx''_\gamma \end{vmatrix} \quad . \tag{72}$$

This is a tensor of rank 3, antisymmetric in all indices.

- The last is an integral over a four-dimensional volume,

$$d^4x \;=\; dx^0 dx^1 dx^2 dx^3 \;\equiv\; cdt\, d\mathcal{V} \quad . \tag{73}$$

The is clearly a scalar due to the compensation between time dilation and length contraction. For this reason, it is often convenient to express 3D integrals as 4D ones with delta functions in the integrands, as needed.

To prove that it is a scalar, i.e. $\underline{d^4x \text{ is a Lorentz invariant}}$, we observe that $d^4x = \mathcal{J} d^4x'$, where the Jacobian for the $x_\mu \to x'_\mu$ transformation, i.e.

$$\mathcal{J} \;=\; \begin{vmatrix} \frac{\partial x_0}{\partial x'_0} & \frac{\partial x_0}{\partial x'_1} & \frac{\partial x_0}{\partial x'_2} & \frac{\partial x_0}{\partial x'_3} \\ \frac{\partial x_1}{\partial x'_0} & \frac{\partial x_1}{\partial x'_1} & \frac{\partial x_1}{\partial x'_2} & \frac{\partial x_1}{\partial x'_3} \\ \frac{\partial x_2}{\partial x'_0} & \frac{\partial x_2}{\partial x'_1} & \frac{\partial x_2}{\partial x'_2} & \frac{\partial x_2}{\partial x'_3} \\ \frac{\partial x_3}{\partial x'_0} & \frac{\partial x_3}{\partial x'_1} & \frac{\partial x_3}{\partial x'_2} & \frac{\partial x_3}{\partial x'_3} \end{vmatrix} \quad . \tag{74}$$

Each of the elements of this determinant is a constant because of the linearity of the form of Lorentz transformations. The determinant can be expressed as a product of simpler determinants corresponding to rotations in two coordinates:[4] *the overall transformation is a sequence of more elementary Lorentz boosts and rotations.* These more elementary rotations in 4D spacetime clearly have determinants equal to unity as all rotations do, whether they involve real or imaginary angles. Thus d^4x is a Lorentz invariant under generalized rotations.

[*Reading Assignment: Pages 20-21 of L&L: types of spacetime integrations*]

[4]This is easily discerned for boost matrices in the (x, t) plane.

2. RELATIVISTIC DYNAMICS

Matthew Baring — Lecture Notes for PHYS 532, Spring 2023

1 Lagrangian and Hamiltonian Mechanics

To set the scene for our consideration of relativistic dynamics, we offer first a brief review of classical Lagrangian and Hamiltonian mechanics. This formalism is based on a **variational principle** that provides a powerful tool for describing dynamics of complex systems of particles. The starting point is that a special quantity called **the action**, S, when minimized, defines the trajectory of a system (or particle) in generalized co-ordinate space ($\mathbf{q}_i, \dot{\mathbf{q}}_i$):

$$\boxed{\delta S \;=\; 0 \quad \text{for} \quad S \;=\; \sum_i \int_{t_a}^{t_b} L(\mathbf{q}_i,\, \dot{\mathbf{q}}_i,\, t)\, dt \quad .} \tag{1}$$

This constrains **the equation of motion** between points a and b at times t_a and t_b. Here L is the **Lagrangian**, which has units of energy, and nominally serves as the kinetic energy (for a free particle). The **action integral** is evaluated for various functional forms for $\mathbf{q}_i$ and $\dot{\mathbf{q}}_i = d\mathbf{q}_i/dt$.

Plot: Sketch a range of paths

Eq. (1) embodies **Hamilton's Principle**, or the **principle of least action** (L&L terminology), which states that *the actual motion occurs when* S *is minimized* and $\delta S = 0$. The immediate consequence of this principle is that one can determine the equation of motion, the **Euler-Lagrange equations**.

$*$ Usually, q_i represents the position of particle i, so that its derivative describes the velocity of the particle.

Consider the simpler case of one set of canonical variables, q and $\dot{q}$. Let $q(t)$ be the variational solution for the actual trajectory. Consider perturbations about this equilibrium solution, $q \to q + \delta q$ and $\dot{q} \to \dot{q} + \delta\dot{q}$. Fix the endpoint values at points a and b, so that no variations arise due to them, thereby fixing $\delta q(t_a) = 0 = \delta q(t_b)$. The variation in the action becomes

$$\delta S \;=\; \int_{t_a}^{t_b} L(q + \delta q,\, \dot{q} + \delta\dot{q},\, t)\, dt - \int_{t_a}^{t_b} L(q,\, \dot{q},\, t)\, dt \quad . \tag{2}$$

A Taylor series expansion of the integrand in the first integral then yields

$$\delta S \;=\; \int_{t_a}^{t_b} \left(\frac{\partial L}{\partial q}\, \delta q + \frac{\partial L}{\partial \dot{q}}\, \delta\dot{q} \right) dt \quad . \tag{3}$$

Since $\delta\dot{q} = d(\delta q)/dt$, we can integrate the second term by parts:

$$\int_{t_a}^{t_b} \frac{\partial L}{\partial \dot{q}} \left(\frac{d}{dt}\, \delta q \right) dt \;=\; \left[\frac{\partial L}{\partial \dot{q}}\, \delta q \right]_{t_a}^{t_b} - \int_{t_a}^{t_b} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q\, dt \tag{4}$$

The first term on the RHS is identically zero since the endpoints are fixed. The variation of the action then takes the form

$$\delta S \;=\; -\int_{t_a}^{t_b} \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} \right\} \delta q\, dt \quad , \tag{5}$$

which can only be zero if the factor in curly braces is zero. Accordingly,

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) \;=\; \frac{\partial L}{\partial \mathbf{q}} \quad ,} \tag{6}$$

which define the **Euler-Lagrange equations** (E-L) of motion; this result has been routinely extended to the more general vector form.

• If L has the dimensions of energy and q the dimensions of length, then each term in the E-L result has the *dimensions of force*. Accordingly, define

$$\mathbf{p} \;\equiv\; \frac{\partial L}{\partial \dot{\mathbf{q}}} \tag{7}$$

to be the **canonical momentum**, conjugate to the space variable $\mathbf{q}$. We can explicitly solve this for $\dot{\mathbf{q}}(\mathbf{p})$.

- This should be familiar from quantum theory. Equations of motion for a system intimately couple distance and momentum, which in a Fourier representation leads to the appearance of complex exponentials of the form $\exp\{i(\mathbf{p}\cdot\mathbf{x})/\hbar\}$. Eventually, the extension of this to four-vector formalism automatically implies additional time-dependent factors $\exp\{-iEt/\hbar\}$.