3.1 Transformation of Velocities

To prepare the way for future considerations of particle dynamics in special relativity, we need to explore the Lorentz transformation of velocities. These are simply derived from our spacetime transformation equations. Let a body be traveling with speed $v_x = dx/dt$ in the K frame, and $v'_x = dx'/dt'$ in the K' frame. We have, again **L&L Sec. 5**

$$x \;=\; \gamma\left(x' + \beta c t'\right) \;,\quad y \;=\; y' \;,\quad z \;=\; z' \;,\quad t \;=\; \gamma\left(t' + \frac{\beta x'}{c}\right) \quad , \tag{29}$$

where $\gamma = 1/\sqrt{1 - V^2/c^2}$ captures the relative velocity V of the two frames, parallel to the x and x' axes. Dividing the spatial equations through by the temporal one, and defining vector velocities by

$$\mathbf{v} \;=\; \frac{d\mathbf{r}}{dt} \quad , \quad \mathbf{v}' \;=\; \frac{d\mathbf{r}'}{dt'} \tag{30}$$

we find

$$v_x \;=\; \frac{v'_x + V}{1 + v'_x V/c^2} \;,\quad v_y \;=\; \frac{v'_y}{\gamma\,(1 + v'_x V/c^2)} \;,\quad v_z \;=\; \frac{v'_z}{\gamma\,(1 + v'_x V/c^2)} \tag{31}$$

as the Lorentz transformation for the three velocity components. This clearly reduces to the familiar Galilean transformation form when $V \ll c$.

$\ast$ Observe that in general, the velocity components transverse to the boost decline since time is dilated: as $V \to c$, $v_y \to 0$ and $v_z \to 0$.

$\ast$ The inverse relations interchange $\mathbf{v}$ and $\mathbf{v}'$ and set $V \to -V$.

$\ast$ In the case of light propagation, $v \to c$, the denominators have important consequences in radiation theory, e.g., for **Čerenkov radiation**.

- Clearly v_x is a monotonically increasing function of v'_x, with the maximum value of $v_x \to c$ asymptotically realized when $v'_x \to c$. Accordingly, the Lorentz transformations embody the principle that all (massive) bodies travel less than the speed of light, as perceived by any observer.

- If one tries to form a kinetic energy formula using $v_x^2 + v_y^2 + v_z^2$, it quickly becomes apparent that the algebra becomes messy. This motivates a more sophisticated representation of spacetime coordinates and velocity, and this will be realized shortly when **four-vectors** are explored.

Suppose now that $v_z = 0$ and set $v_x = v \cos\theta$ and $v_y = v \sin\theta$, so that θ represents the angle of the velocity vector to the boost direction. Similarly define θ' in the K' frame. Then, dividing v_y by v_x, the velocity Lorentz transformations yield

$$\tan\theta \;=\; \frac{v_y}{v_x} \;=\; \frac{v'_y}{\gamma\,(v'_x + V)} \;\equiv\; \frac{v' \sin\theta'}{\gamma(v' \cos\theta' + V)} \quad . \tag{32}$$

This clearly exhibits the property that the velocity vector $\mathbf{v}'$ collapses towards the boost direction $\vec{V}$ due to the motion of the K' frame, when viewed from the K frame, provided that $0 \leq \theta' \leq \pi/2$. For obtuse angles, the opposite behaviour is observed.

∗ This property defines **relativistic beaming** that is commonplace in astrophysical settings. It explains why relativistic outflows emanating from supermassive black holes are so luminous.

For light, we can set $v' \to c$, and write $\beta = V/c$. Then

$$\begin{aligned}
\cos^2\theta \;&=\; \frac{1}{1+\tan^2\theta} \;=\; \frac{\gamma^2(\beta+\cos\theta')^2}{\gamma^2\beta^2 + 2\gamma^2\beta\cos\theta' + \gamma^2\cos^2\theta' + 1 - \cos^2\theta'} \\
&=\; \frac{\gamma^2(\beta+\cos\theta')^2}{\gamma^2 + 2\gamma^2\beta\cos\theta' + \gamma^2\beta^2\cos^2\theta'} \;=\; \frac{(\beta+\cos\theta')^2}{(1+\beta\cos\theta')^2} \quad ,
\end{aligned} \tag{33}$$

using the indentity $\gamma^2\beta^2 + 1 = \gamma^2$, from which one can quickly deduce formulae for the relativistic angular **aberration of light** (direction change):

$$\cos\theta \;=\; \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \quad , \quad \sin\theta \;=\; \frac{\sin\theta'}{\gamma(1+\beta\cos\theta')} \quad . \tag{34}$$

These results can also be obtained directly from the velocity Lorentz transformation. They appear, albeit disguised, in the mathematical structure of the **Klein-Nishina cross section** for Compton scattering.

- This changing of the direction of light when viewed from different inertial reference frames *is a direct consequence of the postulate of the constancy of the speed of light.*

4 Four Vectors in Spacetime

It is time to establish a more sophisticated mathematical formalism for dealing with spacetime manipulations.

Start with the spacetime coordinate position **four-vector** x^μ, defined by **L&L Sec. 6**

$$x^\mu \;=\; (x^0 = ct,\, x^1 = x,\, x^2 = y,\, x^3 = z) \quad . \tag{35}$$

Here the use of Greek indices denotes four-vectors. The **magnitude** of this four-vector is given by the line-element length

$$s^2 \;\equiv\; x_\mu x^\mu \;=\; (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \quad . \tag{36}$$

This, of course, is a Lorentz invariant, and we therefore term it a **scalar**. The product notation resembles the dot product in 4D if we define

$$x_\mu \;=\; (x_0 = ct,\, x_1 = -x,\, x_2 = -y,\, x_3 = -z) \quad . \tag{37}$$

Both x^μ and x_μ are two forms representing the same quantity. We call x^μ the **contravariant** form and x_μ the **covariant** version. The dot product form then obeys the **Einstein summation convention**, namely that

$$x_\mu x^\mu \;\equiv\; x^\mu x_\mu \;\to\; \sum_{\mu=0}^{3} x^\mu x_\mu \quad ; \tag{38}$$

all indices appear twice (μ here) are summed over. This is also termed the **scalar product** of the contravariant and covariant spacetime four-vectors.

The Lorentz transformation has the property that it preserves the magnitude of differential spacetime elements, dx^μ:

$$dx_\mu dx^\mu \;=\; dx'_\nu dx'^\nu \quad \text{with} \quad dx^\mu \;=\; \frac{\partial x^\mu}{\partial x'^\nu}\, dx'^\nu \quad . \tag{39}$$

This constraint is imposed by the fixed value of the speed of light in all inertial reference frames, the *fundamental premise of special relativity*.

For simplicity, consider velocity **boosts** of magnitude βc in the x^1 direction, with $|\beta| < 1$. Given the form of Eq. (36), as was discerned in Sec. 3, a single parameter description of boost invariance is provided by hyperbolic functions, with $x^0 \propto \cosh\psi$ and $x^1 \propto \sinh\psi$, and $\beta = \tanh\psi$. Here $\gamma \equiv 1/\sqrt{1-\beta^2} = \cosh\psi$. One way to facilitate the Lorentz transformation algebra is via a matrix construction. Representing the 4-vectors as column matrices, then the boost equations can be compactly expressed via

$$\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} (x^0)' \\ (x^1)' \\ (x^2)' \\ (x^3)' \end{bmatrix} \quad . \tag{40}$$

The covariant form of this can be quickly cast in terms of row-vectors. It naturally must involve a matrix describing the inverse Lorentz transformation matrix, which sends $\beta \to -\beta$, and this path can be used to establish the Lorentz invariance $x_\mu x^\mu = (x_\mu)'\,(x^\mu)'$.

- We can extend this structure to other four-vectors (4D vectors) and arrive at a Lorentz invariance scalar product if we ascribe to the four-vector the same Lorentz transformation properties that are satisfied by the spacetime coordinates under boosts. This now *becomes a matter of definition*: we <u>define a four-vector</u> A^μ to be a 4D vector that obeys (*contravariant form*)

$$\begin{aligned} A^0 \;&=\; \gamma\Big[(A^0)' + \beta(A^1)'\Big] \quad , \quad A^1 \;=\; \gamma\Big[(A^1)' + \beta(A^0)'\Big] \\ A^2 \;&=\; (A^2)' \quad , \quad A^3 \;=\; (A^3)' \quad . \end{aligned} \tag{41}$$

This applies here specifically for boosts in the x^1 direction, but can be routinely generalized to arbitrary boosts. The *covariant* form is obtained from

$$A_0 \;=\; A^0 \quad , \quad A_1 \;=\; -A^1 \quad , \quad A_2 \;=\; -A^2 \quad , \quad A_3 \;=\; -A^3 \quad , \tag{42}$$

and automatically obeys the Lorentz transformations. Thus,

$$A_\mu A^\mu \;\equiv\; A^\mu A_\mu \;=\; \sum_{\mu=0}^{3} A^\mu A_\mu \;=\; (A')_\mu\,(A')^\mu \tag{43}$$

is a Lorentz invariant (show this algebraically from Eq. 41).

Eq. (43) expresses the square of the magnitude, or **length** of the spacetime 4-vector A^μ. It does not have to be real. If it is real, i.e. $A_\mu A^\mu$ is positive, then this 4-vector is called *timelike*, contrasting $A_\mu A^\mu < 0$ cases where A^μ is termed *spacelike*. A 4-vector that is of zero length is called a **null vector**.

The **inner product** or dot product of two four vectors A^μ and B^μ is simply defined by the summation

$$A_\mu B^\mu \;\equiv\; \sum_{\mu=0}^{3} A_\mu B^\mu \quad , \tag{44}$$

and can be positive or negative.

- The matrix formulation affords a routine protocol to efficiently and accurately compute Lorentz transformation of four-vectors.

- The Lorentz transformation matrix is just a Jacobian:

$$\mathbf{\Lambda}^\mu{}_\nu \;\equiv\; \frac{\partial x^\mu}{\partial x'^\nu} \;=\; \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;=\; \begin{bmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{bmatrix} \quad . \tag{45}$$

Observe that the first index, μ, in $\mathbf{\Lambda}^\mu{}_\nu$ corresponds to the matrix <u>row</u> number, while the second, ν, marks the <u>column</u> number. This is established via the construction in Eq. (40). One can then apply this matrix protocol to sequential Lorentz boosts since the chain rule for differentiation is operative for the two coordinate transformation functions $x^\mu(x'^\sigma)$ and $x'^\sigma(x''^\nu)$:

$$dx^\mu \;=\; \frac{\partial x^\mu}{\partial x'^\nu}\, dx'^\nu \;=\; \frac{\partial x^\mu}{\partial x'^\sigma}\, \frac{\partial x'^\sigma}{\partial x''^\nu}\, dx''^\nu \tag{46}$$

(remember that the Einstein summation convention applies here) which corresponds to a composite boost:

$$\mathbf{\Lambda}_c \;\to\; \mathbf{\Lambda}^\mu_\sigma\, \mathbf{\Lambda}^\sigma_\nu \quad . \tag{47}$$

This matrix manipulation technique appears extremely useful for 4-vectors: we will find it also so for tensors, our incipient focus.

• Observe that the Lorentz transformation matrix $\mathbf{\Lambda}^{\mu}_{\nu}$ is *symmetric* and *has a determinant equal to unity.* This unit determinant is a general property of boost matrices, because all such boosts constitute a sequence of rotations in spacetime, and all rotations preserve volume elements and solid angles.

• L&L use Roman letters for the 4 spacetime indices and Greek for the purely space ones, the opposite of the widely-used convention employed here.

• Four-vectors will be useful in expressing spacetime quantities that are related and combine to define Lorentz invariants. Examples include the *four-momentum,* $p^{\mu} = (E, \mathbf{p})$ and the *four-vector potential* $A^{\mu} = (\phi, \mathbf{A})$.

4.1 Four-velocity and Four-acceleration

To serve as an immediate example of a four-vector, we generalize the ordinary 3D velocity $\mathbf{v}$ to define the **four-velocity** u^{μ} of a particle via **L&L Sec. 7**

$$u^{\mu} \;=\; \frac{dx^{\mu}}{ds} \quad \text{for} \quad ds = c\,dt\,\sqrt{1-\frac{v^2}{c^2}} \;=\; cd\tau \quad . \tag{48}$$

Here $\tau \equiv t'$ is the proper time, evaluated in the K' frame in which the particle is at rest. Therefore the Lorentz factor of the particle is

$$\gamma \;=\; \frac{1}{\sqrt{1-v^2/c^2}} \;\equiv\; \cosh\psi \quad , \tag{49}$$

so that $ds = cdt/\gamma$. The x-component of the four velocity is

$$u^1 \;=\; \frac{dx}{c\,dt\,\sqrt{1-v^2/c^2}} \;=\; \gamma\,\frac{v_x}{c} \quad . \tag{50}$$

Here the v pertains to the *total* particle speed, not just v_x , since the time dilation factor is pertinent to the boost to the particle rest frame. Similar results ensue for the other velocity components. In compact form,

$$u^{\mu} \;=\; \gamma\Big(1,\, \beta\,\hat{v}\Big) \;\equiv\; \gamma\big(1,\, \boldsymbol{\beta}\big) \quad , \quad \beta \;=\; \frac{v}{c} \quad , \tag{51}$$

with $\hat{v} = \mathbf{v}/v$ as the unit vector in the direction of the velocity of the particle, as measured in the observer's frame.

• Since $dx_\alpha dx^\alpha = ds^2$, one simply has

$$u_\mu u^\mu \;\equiv\; \frac{dx_\mu dx^\mu}{ds^2} \;=\; 1 \quad , \tag{52}$$

i.e. the 4-velocity always has unit length that is therefore a Lorentz invariant. This just expresses the constancy of the speed of light in yet another form. Consequently, *the components of the four-velocity are not independent.*

$\ast$ The "normalization condition" $u_\mu u^\mu = 1$ extends also to general relativity, where one generalizes the covariant form $u_\mu = g_{\mu\nu} u^\nu$ for $g_{\mu\nu} \neq \eta_{\mu\nu}$.

It is now routine to extend the definition of acceleration also to define a second derivative of the spacetime 4-vector as the **four-acceleration**:

$$a^\mu \;=\; \frac{d^2 x^\mu}{ds^2} \;\equiv\; \frac{du^\mu}{ds} \quad . \tag{53}$$

Note that L&L use w^μ for 4-acceleration. This quantity will prove useful for the consideration of radiation by accelerating relativistic charges.

• A routine differentiation of the four-velocity normalization condition in Eq. (52) with respect to path length s promptly yields

$$\begin{aligned} 0 \;=\; \frac{1}{2}\frac{d}{ds}\Big(u_\mu u^\mu \Big) \;&=\; \frac{1}{2}\left(\frac{du_\mu}{ds}\, u^\mu + u_\mu\, \frac{du^\mu}{ds} \right) \\ &=\; \frac{1}{2}\Big(a_\mu u^\mu + u_\mu a^\mu \Big) \;=\; u_\mu a^\mu \quad . \end{aligned} \tag{54}$$

Thus, *the four-velocity and four-acceleration are always orthogonal.* This result can prove useful when integrating over particle trajectories, and again is a consequence of the constancy of the speed of light.

• L&L outline the interesting problem of motion under a constant, uniform acceleration. The familiar Galilean result $x \propto at^2$ must fail miserably as c is approached, and at asymptotically large times, $x \approx ct$ must result.

[*Reading Assignment: Uniform acceleration problem – Section 7, page 24.*]

5 Tensors in Spacetime

We now extend this algebra to define a **4-tensor of rank 2** to be a set of 16 quantities $T^{\mu\nu}$, which under coordinate transformations maps like a product of two 4-vectors (which are 4-tensors of rank one). One can similarly define tensors of higher rank, i.e. with more indices. A rank 2 tensor looks like

$$\begin{bmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{bmatrix} \cdot \begin{bmatrix} B^0 & B^1 & B^2 & B^3 \end{bmatrix} \tag{55}$$

This maps out to a 4×4 matrix, and this will be the natural representation that will be adopted below for tensors suited to electrodynamics theory.

There are three alternative forms describing a second-rank 4-tensor that extend the two forms we have encountered for 4-vectors:

$$\begin{array}{rcl} T^{\mu\nu} & : & \text{contravariant} \\ T^{\mu}_{\nu} \;\text{ or }\; T_{\nu}^{\;\mu} & : & \text{mixed} \\ T_{\mu\nu} & : & \text{covariant} \quad . \end{array} \tag{56}$$

The connection between the different types of components is determined by the general rule: *raising or lowering a space index* $(1, 2, 3)$ *changes the sign of the component, while raising or lowering the time index* (0) *does not*. So,

$$T_{00} \;=\; T^{00} \quad , \quad T_{01} \;=\; -T^{01} \quad , \quad T_{11} \;=\; T^{11} \quad , \quad \text{etc.} \tag{57}$$

Technically, there are two types of mixed components. However, in practice, these are often identical due to the diagonality of the $\eta_{\alpha\beta}$ Minkowski matrix, and so in such cases we represent them without detailed attention to which index is raised or lowered. We have the relations

$$T^0_0 \;=\; T^{00} \quad , \quad T^1_0 \;=\; T^{01} \quad , \quad T^0_1 \;=\; -T^{01} \quad , \quad T^1_1 \;=\; -T^{11} \quad , \quad \text{etc.} \tag{58}$$

A tensor is **symmetric** if $T^{\mu\nu} = T^{\nu\mu}$, or is **antisymmetric** if $T^{\mu\nu} = -T^{\nu\mu}$. Obviously, the diagonal elements of an antisymmetric tensor are all zero.

* For a symmetric tensor, $T^{\mu}_{\nu} = T_{\nu}^{\;\mu}$, so we just write it T^{μ}_{ν}.