## 2.1 Proper time

We now have all the tools to assess how clocks appear to tick for different observers. Park a moving clock in the $K'$ frame so that it is stationary therein, and ticks at time intervals $dt'$. In the $K$ frame it is moving with speed $V$, covering a distance $dl = \sqrt{dx^2 + dy^2 + dz^2}$ in time interval $dt$. The quantities in the two frames are related via

**L&L Sec. 3**

$$ds^2 \; = \; c^2 dt^2 - dl^2 \; = \; c^2 (dt')^2 \quad . \tag{13}$$

The measured time intervals $dt$ obtained by an observer in frame $K$ then satisfy

$$dt' \; = \; \frac{ds}{c} \; = \; dt \sqrt{1 - \frac{V^2}{c^2}} \quad , \quad V \; = \; \frac{dl}{dt} \quad . \tag{14}$$

This implies that the $K$ frame observer sees **time dilation**, i.e. the clock appears to slow down since $dt > dt'$. This can also be written

$$dt \; = \; \gamma \, dt' \quad , \quad \gamma \; = \; 1 \bigg/ \sqrt{1 - \frac{V^2}{c^2}} \quad . \tag{15}$$

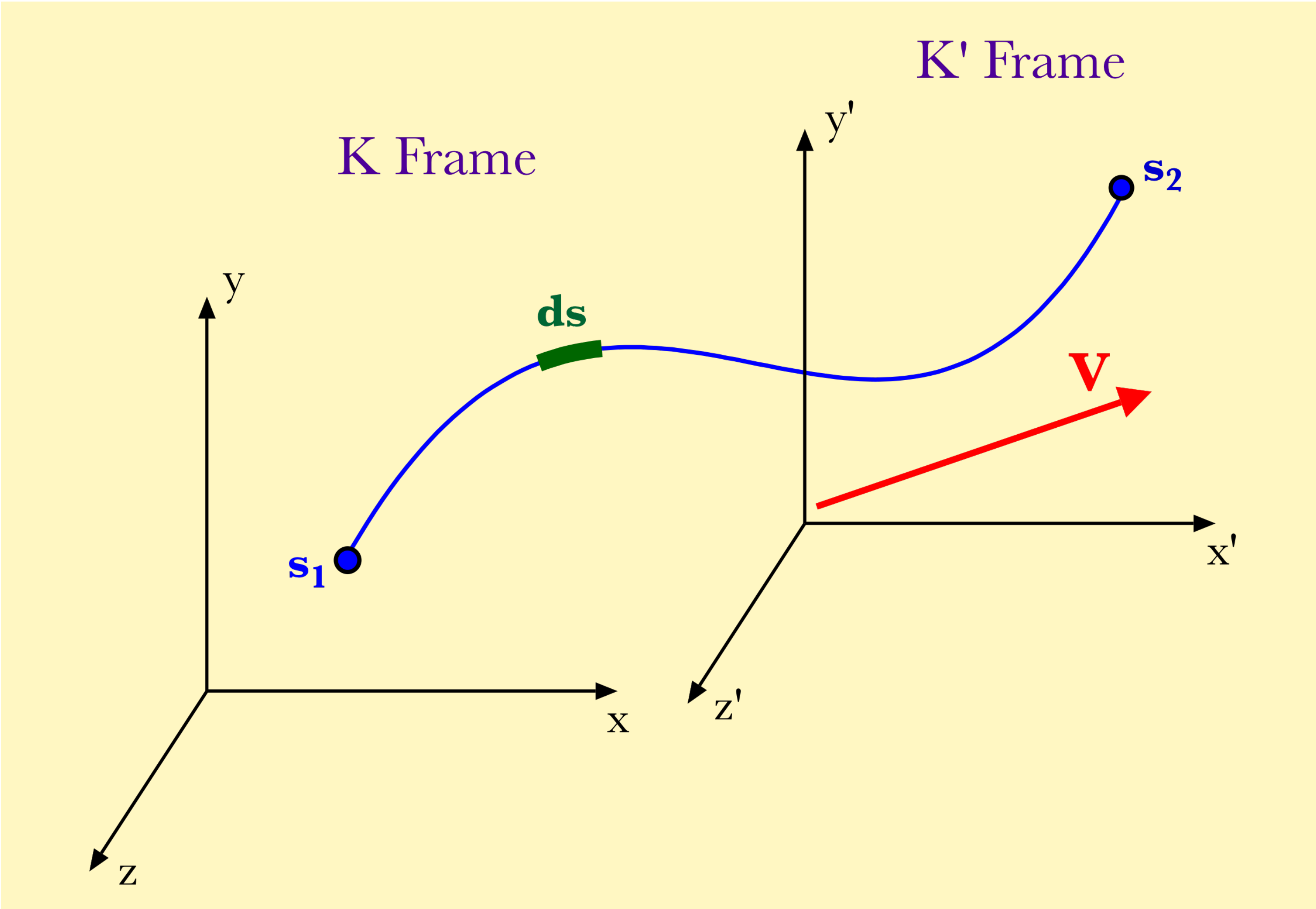
Here $\gamma \geq 1$ is the **Lorentz factor** of the relativistic frame transformation or **boost**. Integrating over finite intervals, one has

$$t_2' - t_1' \; = \; \int_{t_1}^{t_2} dt \sqrt{1 - \frac{V^2}{c^2}} \quad , \quad V \; = \; V(t) \quad . \tag{16}$$

This interval is unique among all inertial frames, since it pertains to the only frame in which the clock is at rest. It is therefore called the **proper time** of the clock. We all experience the same proper time, even if we are moving.

- Note that by symmetry, since $-V$ is the relative speed/velocity of the $K$ frame to the $K'$ frame, an observer in the $K'$ frame will determine that an identical clock that is stationary in the $K$ frame will tick slower by precisely the same time dilation factor, $\sqrt{1 - V^2/c^2}$.

# Two Inertial Frames in Relative Motion



• There is no conflict here. To actually measure the clock ticks in another frame in relative motion requires the propagation of light from different positions. How these positions are established in the two frames when circumstances are reversed is not a convolution of identical algebraic dilation factors. See L&L for an explanation.

$*$ If we were to take two clocks initially in the $K$ frame, and move one to the $K'$ frame for a while, and then return it to the $K$ frame, one would not necessarily expect the the two clocks to measure the same time, since there have been episodes of non-inertial world-line paths for the accelerated clock. This violates the conditions for special relativity and can be resolved by general relativity.

**Plot:** World line viewed in two inertial frames.

• Yet we can make an important inference. Along the **world line** or spacetime path of a clock, the time interval it reads between two spacetime points $a$ and $b$ is given by

$$\frac{1}{c}\int_a^b ds \quad . \tag{17}$$

If the clock is at rest for the duration of this spacetime "transit," then this defines the proper time interval. If instead the clock moves with a non-uniform motion starting at $a$ and ends up at $b$ (in a sequence of timelike intervals: illustrate this relative to the light cone), as we know that the clock at rest indicates greater time intervals than a moving one, we deduce that

$$\frac{1}{c}\int_a^b ds \tag{18}$$

*is maximized* when taking a straight world-line in the 4D pseudo-Euclidean spacetime, termed **Minkowski spacetime**. All other paths incur *negative* contributions from $-(dx^2 + dy^2 + dz^2)$ that reduce $ds^2$.

# 3 The Lorentz Transformation

It is now time to form the precise relationship between spacetime coordinates in two inertial frames in relative motion. These should reduce, in the limit of small boost speeds $V \ll c$, to the **Galilean transformation**: **L&L Sec. 4**

$$x \;=\; x' + Vt \quad , \quad y \;=\; y' \quad , \quad z \;=\; z' \quad , \quad t \;=\; t' \quad , \tag{19}$$

presuming a boost along the $x$-axis from frame $K$ to frame $K'$. The central property is that *this transformation must leave all distances unchanged* in the 4D $x, y, z, ct$ space. To find the clue to the expedient path, we recast the differential world line element equation as

$$ds^2 \;=\; (c\,dt)^2 + (i\,dx)^2 + (i\,dy)^2 + (i\,dz)^2 \quad . \tag{20}$$

Now all contributions resemble Euclidean form. Accordingly, by analogy, *the only transformations that preserve lengths are rotations* in our 4D hyperspace.

Every rotation in ordinary 3D space can be distilled into a sequence of three rotations in the $x-y$, $y-z$ and $z-x$ planes. By extension, we can distill our generalizes spacetime rotation into a sequence of six ( $= {}^4C_2/2$ ) rotations in the $x-y$, $y-z$, $z-x$, $x-t$, $y-t$ and $z-t$ planes. For simplicity, we will consider boosts parallel to the $x$ and $x'$ axes, so that only one rotation need be considered. We then impose

$$c^2t^2 - x^2 \;=\; c^2(t')^2 - (x')^2 \quad . \tag{21}$$

This 2D Lorentz invariance criterion can be satisfied only by linear transformations of the mathematical form

$$x \;=\; x' \cosh\psi + ct' \sinh\psi \quad , \quad ct \;=\; ct' \cosh\psi + x' \sinh\psi \quad . \tag{22}$$

It is almost trivial to conclude that these relations satisfy Eq. (21). The appearance of hyperbolic functions is a direct consequence of the pseudo-Euclidean nature of spacetime: a substitution $\psi \to i\psi$ would generate trigonometric functions appropriate to rotations in true Euclidean spaces. Observe that $\psi = \psi(V)$ is the only possible mathematical dependence.

To obtain the precise functional dependence for $\psi(V)$, one considers an object at rest in the $K'$ frame. Then $x' = 0$ at all $t'$, and hence

$$x \;=\; ct' \sinh\psi \quad , \quad ct \;=\; ct' \cosh\psi \quad \Rightarrow \quad \frac{V}{c} \;\equiv\; \frac{x}{ct} \;=\; \tanh\psi \quad . \tag{23}$$

We have the solution $\psi = \tanh^{-1}(V/c)$, so that if we set $\beta = V/c$,

$$\cosh\psi \;=\; \gamma \quad , \quad \sinh\psi \;=\; \gamma\beta \quad \text{for} \quad \gamma \;=\; \frac{1}{\sqrt{1-\beta^2}} \quad . \tag{24}$$

The transformation parameter $\psi$ is called the **rapidity**. The overall set of transformation equations is

$$\boxed{x \;=\; \gamma\Big(x' + \beta ct'\Big) \,, \quad y \;=\; y' \,, \quad z \;=\; z' \,, \quad t \;=\; \gamma\left(t' + \frac{\beta x'}{c}\right) \quad .} \tag{25}$$

This is the form of a **Lorentz transformation**, which in the limit of small boost speeds $\beta \ll 1$ (for which $\gamma \to 1$), reduces to the familiar Galilean transformation. The inverse transformation is simply obtained by $\beta \to -\beta$.

$*$ Lorentz transformations in general directions can be more easily deduced by rotating the coordinate systems first, and employing this form.

The reason underpinning this assertion is that the result of two consecutive Lorentz transformations depends on the order in which they are performed: *in general, Lorentz transformations are not commutative operators.* This means that the result of a boost of $\vec{V}_1$ followed by a boost by $\vec{V}_2$ differs from that from $\vec{V}_2$ followed by $\vec{V}_1$. The exception is when the two boosts are parallel, $\vec{V}_1 \times \vec{V}_2 = \mathbf{0}$.

$*$ That this should be true can be inferred from consideration of the ordering of successive rotations in 3D Euclidean space.

$*$ This non-commutativity distinguishes Lorentz transformations from Galilean ones where the order does not matter for general vector directions.

It is now appropriate to illustrate two immediate consequences of the Lorentz transformations: **length contraction** and **time dilation**.

• Consider a rod at rest in the $K'$ system, parallel to its $x'$ axis (L&L places it at rest in the $K$ system; but we need to be consistent with above). Let its two endpoints be $x'_1$ and $x'_2$, so that its length in the $K'$ frame is $\Delta x' = x'_2 - x'_1$, which is termed the **proper length**. The endpoints as discerned in the $K$ frame are $x_1$ and $x_2$, and these must be determined at the same time $t$ therein. Hence, the inverse Lorentz transformation must be applied and we have

$$x'_1 \;=\; \gamma\Big(x_1 - \beta ct\Big) \quad , \quad x'_2 \;=\; \gamma\Big(x_2 - \beta ct\Big) \quad . \tag{26}$$

Accordingly, the length of the rod as measured in the $K$ frame is

$$\Delta x \;\equiv\; x_2 - x_1 \;=\; \frac{\Delta x'}{\gamma} \quad , \tag{27}$$

and is *shorter* than the proper length: motion of a moving body **contracts its length** along the direction of motion.

$*$ Dimensions transverse to the boost are not length-contracted, so volumes are "compressed" by only one power of the Lorentz factor: $\mathcal{V} = \mathcal{V}'/\gamma$.

$*$ Direct measurements of this prediction of special relativity have not been possible, because macroscopic bodies do not move very fast. Only subatomic particles do, and these are not amenable to such tests.

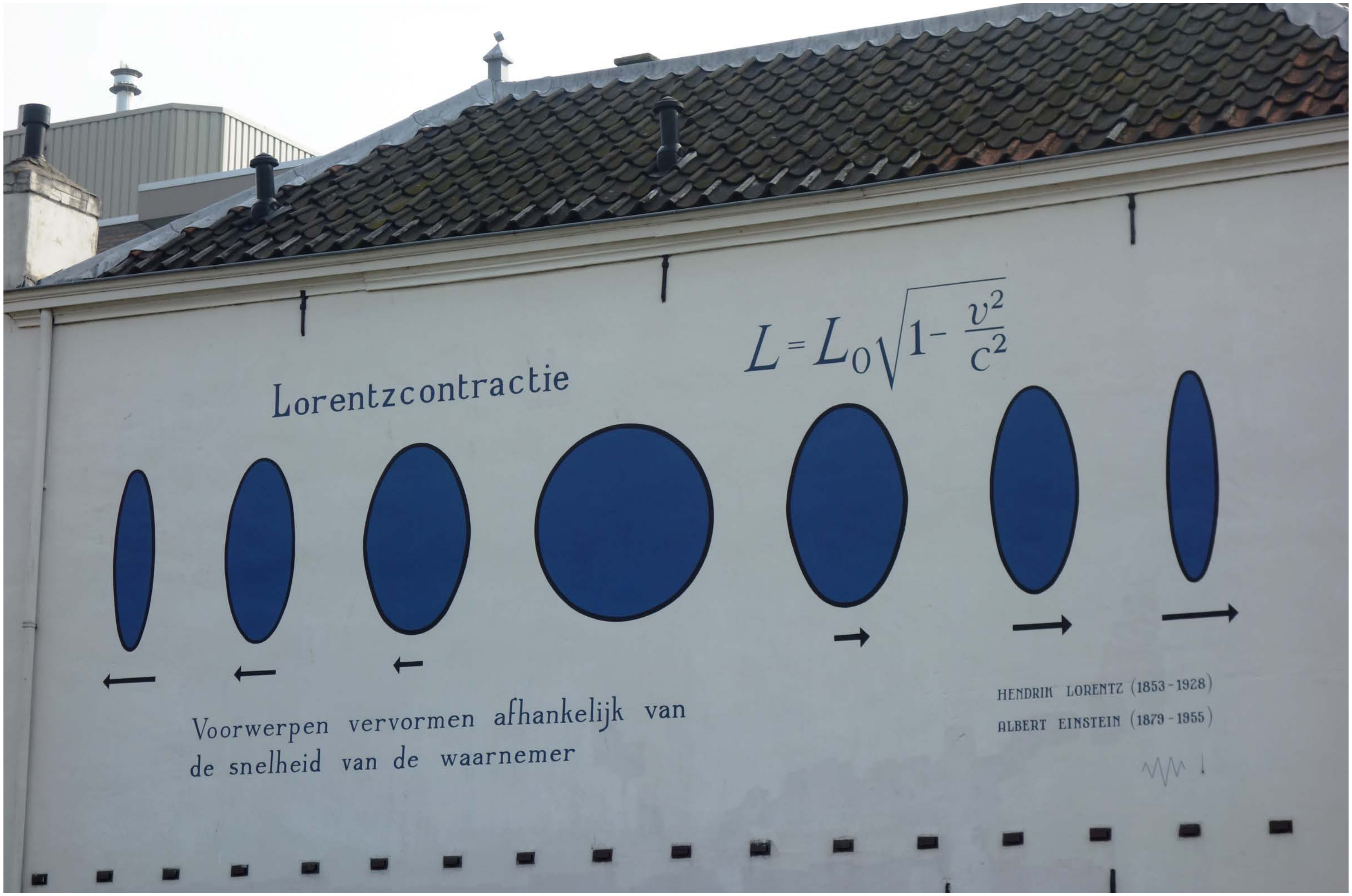
**Plot:** Mural honoring Lorentz's legacy in Leiden

• On the other hand, **time dilation** is measurable, for example through nuclear interaction and elementary particle decay channels. The effect emerges naturally from the Lorentz transformation. Again, our clock is at rest in the $K'$ system, and the proper time between two ticks is $\Delta t' = t'_2 - t'_1$. The times $t_1$ and $t_2$ of the ticks in the $K$ frame are then given by

$$t_1 \;=\; \gamma\left(t'_1 + \frac{\beta x'_1}{c}\right) \quad , \quad t_2 \;=\; \gamma\left(t'_2 + \frac{\beta x'_2}{c}\right) \tag{28}$$

From this one quickly deduces that $\Delta t \equiv t_2 - t_1 = \gamma \Delta t'$, i.e. that time for a moving body is dilated.

# A Leiden Mural on Lorentz Transformations




- Public recognition of Hendrik Lorentz's contributions to special relativity: a length contraction mural in Leiden, The Netherlands.

## 3.1 Transformation of Velocities

To prepare the way for future considerations of particle dynamics in special relativity, we need to explore the Lorentz transformation of velocities. These are simply derived from our spacetime transformation equations. Let a body be traveling with speed $v_x = dx/dt$ in the $K$ frame, and $v'_x = dx'/dt'$ in the $K'$ frame. We have, again **L&L Sec. 5**

$$x \;=\; \gamma\left(x' + \beta c t'\right) \;,\quad y \;=\; y' \;,\quad z \;=\; z' \;,\quad t \;=\; \gamma\left(t' + \frac{\beta x'}{c}\right) \quad , \qquad (29)$$

where $\gamma = 1/\sqrt{1 - V^2/c^2}$ captures the relative velocity $V$ of the two frames, parallel to the $x$ and $x'$ axes. Dividing the spatial equations through by the temporal one, and defining vector velocities by

$$\mathbf{v} \;=\; \frac{d\mathbf{r}}{dt} \quad , \quad \mathbf{v}' \;=\; \frac{d\mathbf{r}'}{dt'} \qquad (30)$$

we find

$$v_x \;=\; \frac{v'_x + V}{1 + v'_x V/c^2} \;,\quad v_y \;=\; \frac{v'_y}{\gamma\,(1 + v'_x V/c^2)} \;,\quad v_z \;=\; \frac{v'_z}{\gamma\,(1 + v'_x V/c^2)} \qquad (31)$$

as the Lorentz transformation for the three velocity components. This clearly reduces to the familiar Galilean transformation form when $V \ll c$.

$*$ Observe that in general, the velocity components transverse to the boost decline since time is dilated: as $V \to c$, $v_y \to 0$ and $v_z \to 0$.

$*$ The inverse relations interchange $\mathbf{v}$ and $\mathbf{v}'$ and set $V \to -V$.

$*$ In the case of light propagation, $v \to c$, the denominators have important consequences in radiation theory, e.g., for **Čerenkov radiation**.

- Clearly $v_x$ is a monotonically increasing function of $v'_x$, with the maximum value of $v_x \to c$ asymptotically realized when $v'_x \to c$. Accordingly, the Lorentz transformations embody the principle that all (massive) bodies travel less than the speed of light, as perceived by any observer.

- If one tries to form a kinetic energy formula using $v_x^2 + v_y^2 + v_z^2$, it quickly becomes apparent that the algebra becomes messy. This motivates a more sophisticated representation of spacetime coordinates and velocity, and this will be realized shortly when **four-vectors** are explored.