HOMEWORK PROBLEMS 8
PHYS 516, Fall 2022

PDEs, Green's Functions and Integral Equations
Due date: Friday, 12/2/22 — 9pm

1. (a) Derive the infinite series solution for Laplace's equation

$$\frac{\partial^2 \Phi(x,\, y)}{\partial x^2} + \frac{\partial^2 \Phi(x,\, y)}{\partial y^2} \;=\; 0 \tag{1}$$

with the Dirichlet boundary conditions $\Phi(x=0,\, y) = 1\,,\ \Phi(x=1,\, y) = -1$ and $\Phi(x,\, y=0) = 0 = \Phi(x,\, y=1)\,.$

(b) Write a code that will numerically sum this series to an arbitrary number of terms. Display this solution in 3D for a 50×50 equally-spaced (x, y) grid, for the specific cases of sums to $11\,,\ 21$ and 51 terms. You can use the Mathematica plotting function `Plot3D[]`, or some other graphical tool.

In your plot, do you detect any evidence for the oscillation incurred (known as *Gibb's phenomenon*) when the Fourier series is truncated at a finite number of terms? Estimate how many terms in the sum are necessary to obtain about 1% accuracy in the solution over most of parameter space.

[28 points credit]

2. Arfken & Weber Problem 10.5.2. Find the Green's functions for

$$\mathcal{L}y(x) \;=\; \frac{d^2y(x)}{dx^2} + y(x) \quad , \quad \begin{cases} y(0) \;=\; 0\,, \\ y'(1) \;=\; 0\,, \end{cases} \tag{2}$$

and

$$\mathcal{L}y(x) \;=\; \frac{d^2y(x)}{dx^2} - y(x) \quad , \quad y(x) \text{ is finite for } -\infty \;<\; x \;<\; \infty \quad . \tag{3}$$

Note that Arfken & Weber uses a different sign convention for the Green's function from that used in lectures.

• *Hint:* the differential equation that the Green's function G satisfies may guide you to appropriately normalize G.

[28 points credit]

3. Arfken & Weber Problem 16.3.9. Solve the Volterra equation

$$\phi(x) \;=\; 1 + \lambda^2 \int_0^x (x-t)\,\phi(t)\,dt \tag{4}$$

by each of the following methods:

- reduction to an ODE, finding the boundary conditions;
- the Neumann series;
- the use of Laplace transforms.

[22 points credit]

4. Arfken & Weber Problems 16.2.3 and 16.2.4.

(a) Show that the solution of a Volterra equation of the second kind

$$\phi(x) \;=\; f(x) + \lambda \int_0^x k(x-t)\,\phi(t)\,dt \tag{5}$$

with the specialized kernel $K(x,\,t) = k(x-t)$ has the solution

$$\phi(x) \;=\; \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{xs}\,\frac{F(s)\,ds}{1-\lambda\,K(s)} \quad , \tag{6}$$

assuming that the required transforms exist.

(b) Use the Laplace transform solution of part (a) to solve the two integral equations

$$\begin{aligned} \phi(x) &= x + \int_0^x (t - x)\, \phi(t)\, dt \quad , \\ \phi(x) &= x - \int_0^x (t - x)\, \phi(t)\, dt \quad . \end{aligned} \tag{7}$$

Check your results by substituting back into the original equations.

[22 points credit]