3 Neumann Series and Separable Kernels

Two prominent techniques that we outline here are the perturbation theory Neumann series method, and the discrete matrix theory approach for separable kernels. Each has their domain of choice, though their efficacy can be comparable for simple problems. **A&W Sec. 16.3**

3.1 Neumann Series Method

The Neumann series technique is usually applied to Fredholm equations with fixed integration limits, though it can be applied to Volterra equations.

Example 5: To illustrate the Neumann technique, consider the equation

$$\phi(x) \;=\; x + \lambda \int_{-1}^{1} (t - x)\, \phi(t)\, dt \quad . \tag{34}$$

The solution is obviously of the mathematical form

$$\phi(x) \;=\; \alpha x + \beta \quad . \tag{35}$$

If we try the test solution $\phi_0(x) = x$ on the RHS, the exact solution for $\lambda \to 0$, then the first iteration yields

$$\phi_1(x) \;=\; x + \lambda \int_{-1}^{1} (t - x)\, t\, dt \;=\; x + \frac{2\lambda}{3} \quad . \tag{36}$$

The next two iterations are simply obtained:

$$\begin{aligned} \phi_2(x) \;&=\; x + \lambda \int_{-1}^{1} (t - x) \left\{ t + \frac{2\lambda}{3} \right\} dt \;=\; x \left(1 - \frac{4\lambda^2}{3} \right) + \frac{2\lambda}{3} \\ \phi_3(x) \;&=\; x + \lambda \int_{-1}^{1} (t - x)\, \phi_2(t)\, dt \;=\; x \left(1 - \frac{4\lambda^2}{3} \right) + \frac{2\lambda}{3} \left(1 - \frac{4\lambda^2}{3} \right) \end{aligned} \tag{37}$$

The process can be continued, and the even iterations assume the form

$$\phi_{2n} \;=\; x \left\{ 1 + \sum_{s=1}^{n} \Bigl(-\frac{4\lambda^2}{3} \Bigr)^s \right\} - \frac{1}{2\lambda} \sum_{s=1}^{n} \Bigl(-\frac{4\lambda^2}{3} \Bigr)^s \quad , \tag{38}$$

a result that can be demonstrated by induction. The two geometric series can be summed in the limit $n \to \infty$

$$\sum_{s=1}^{\infty}\Big(-\frac{4\lambda^2}{3}\Big)^s \;=\; -\frac{4\lambda^2}{3+4\lambda^2} \tag{39}$$

to derive the convergent solution:

$$\phi(x) \;=\; \lim_{n\to\infty}\phi_{2n}(x) \;=\; \frac{3x+2\lambda}{3+4\lambda^2} \quad . \tag{40}$$

If one inserts Eq. (35) directly into the integral equation, one derives

$$\alpha x + \beta \;=\; x(1-2\beta\lambda) + \frac{2}{3}\,\lambda\alpha \tag{41}$$

so that

$$\alpha \;=\; \frac{3}{3+4\lambda^2} \quad , \quad \beta \;=\; \frac{2\lambda}{3+4\lambda^2} \quad , \tag{42}$$

and the same solution results.

Example 6: To complicate things slightly, consider the Fredholm equation

$$\phi(x) \;=\; \sin x + \lambda\int_0^{2\pi}(t-\sin x)\,\phi(t)\,dt \quad . \tag{43}$$

This too has an obvious form for the solution, namely $\phi(x) = \alpha\sin x + \beta$. Therefore, try the test solution $\phi_0(x) = \sin x$ on the RHS, the exact solution for $\lambda \to 0$. The first iteration then yields

$$\phi_1(x) \;=\; \sin x + \lambda\int_0^{2\pi}(t-\sin x)\,\sin t\,dt \;=\; \sin x - 2\pi\lambda \quad , \tag{44}$$

leading to

$$\phi_2(x) \;=\; (1+4\pi^2\lambda^2)\,\sin x - 2\pi\lambda(1+4\pi^2\lambda) \quad . \tag{45}$$

The ensuing algebra is a little more involved, and after summing over the two pertinent geometric series, one eventually arrives at the solution

$$\phi(x) \;=\; \frac{(2\pi^2\lambda-1)\,\sin x + 2\pi\lambda}{4\pi^2\lambda^2+2\pi^2\lambda-1} \quad . \tag{46}$$

The essence of the Neumann technique is iterative, generating an infinite series that is hopefully convergent. Thus for a general Fredholm equation of the second kind, with $f(x) \neq 0$,

$$\phi(x) \;=\; f(x) + \lambda \int_a^b K(x,\, t)\, \phi(t)\, dt \quad , \tag{47}$$

the first three iterations in the $\phi(x) \approx \phi_n(x)$ sequence are

$$\begin{aligned}
\phi_0(x) &= f(x) \quad , \\
\phi_1(x) &= f(x) + \lambda \int_a^b K(x,\, t)\, f(t)\, dt \quad , \qquad (48) \\
\phi_2(x) &= f(x) + \lambda \int_a^b K(x,\, t_1)\, f(t_1)\, dt_1 \\
&\qquad + \lambda^2 \int_a^b \int_a^b K(x,\, t_1) K(t_1,\, t_2)\, f(t_2)\, dt_2\, dt_1 \quad . \qquad (49)
\end{aligned}$$

The full sequence develops the series

$$\phi_n(x) \;=\; \sum_{m=0}^{n} \lambda^m u_m(x) \quad , \tag{50}$$

for

$$u_m(x) \;=\; \int_a^b \int_a^b \cdots \int_a^b K(x,\, t_1) K(t_1,\, t_2) \ldots K(t_{m-1},\, t_m)\, f(t_m)\, dt_m \ldots dt_1 \;. \tag{51}$$

This type of construction is the essence of perturbation theory in quantum mechanics. If $|f|_{\max}$ and $|K|_{\max}$ are respectively the maximum values of $|f|$ and $|K|$ on the interval $[a,\, b]$ spanning both variables, then one can guarantee convergence of the Neumann series if

$$|\lambda| \cdot |K|_{\max} \cdot |b - a| \;<\; 1 \quad , \tag{52}$$

using the Cauchy ratio test. This is sufficient, yet sometimes not necessary.

• For Example 6 above, this Cauchy convergence criterion translates to $2\pi\lambda < 1$, which would be inferred from the geometric series construction.

3.2 Separable Kernels

The special case of Fredholm integral equations with kernels that are separable in their two arguments presents a useful path to solution as a matrix problem. These constitute kernels of the form

$$K(x,\, t) \;=\; \sum_{j=1}^{n} M_j(x)\, N_j(t) \quad , \tag{53}$$

where n is a positive integer. The sum is therefore finite. The ensuing technique requires modification for cases where the series is infinite, which can be workable in select cases. With this form for the kernel, the variable dependence can be explicitly isolated as follows. We have

$$\phi(x) \;=\; f(x) + \lambda \sum_{j=1}^{n} M_j(x) \int_a^b N_j(t)\, \phi(t)\, dt \quad . \tag{54}$$

Then one can define a vector $\mathbf{n} = \{n_j\}$ whose components are coefficients of a dot product on the RHS of this equation:

$$n_j \;=\; \int_a^b N_j(t)\, \phi(t)\, dt \quad . \tag{55}$$

The mathematical form of the solution for $\phi(x)$ is automatically constrained:

$$\phi(x) \;=\; f(x) + \lambda \sum_{j=1}^{n} n_j\, M_j(x) \quad . \tag{56}$$

The integrals in Eq. (55) are routinely determined using Eq. (56):

$$n_i \;=\; f_i + \lambda \sum_{j=1}^{n} K_{ij} n_j \quad , \tag{57}$$

where

$$f_i \;=\; \int_a^b N_i(t)\, f(t)\, dt \quad \text{and} \quad K_{ij} \;=\; \int_a^b N_i(t)\, M_j(t)\, dt \quad . \tag{58}$$

Accordingly, one has a vector $\mathbf{f} = \{f_i\}$ and a matrix $\mathcal{K} = \{K_{ij}\}$, with the solution path defined by a matrix problem.

Another framing is that this system defines a set of simultaneous equations to be solved. In the matrix notation, for an identity matrix I,

$$\mathbf{f} \;=\; \mathbf{n} - \lambda \mathcal{K}\, \mathbf{n} \quad \Rightarrow \quad \mathbf{n} \;=\; \Big(I - \lambda \mathcal{K}\Big)^{-1} \mathbf{f} \quad . \tag{59}$$

For inhomogeneous equations with $\mathbf{f} \neq \mathbf{0}$, it is possible to find a solution provided that λ satisfies $|I - \lambda \mathcal{K}| \neq 0$.

• Homogeneous equations with $\mathbf{f} = \mathbf{0}$ present a different character to the solution and the path for obtaining such. In this case, we call

$$|I - \lambda \mathcal{K}| \;=\; 0 \tag{60}$$

the **secular equation** for the homogeneous Fredholm equation. It must be satisfied in order to generate a solution to Eq. (54), in which case the system of simultaneous equations for the coefficients is redundant in some way. The secular equation then defines a select group of **eigenvalues** λ_k and **eigensolutions** ϕ_k for which a viable solution is realized. Otherwise no solution is possible.

Example 7: Consider first the inhomogeneous Fredholm equation

$$u(x) \;=\; e^x + \lambda\, C_u\, x \quad , \quad C_u \;=\; \int_0^1 t\, u(t)\, dt \quad . \tag{61}$$

This has a kernel $K(x,\, t) = xt$, which is separable. Multiplying by x and integrating over $0 \leq x \leq 1$ yields

$$C_u \;\equiv\; \int_0^1 x\, u(x)\, dx \;=\; 1 + \frac{\lambda}{3}\, C_u \quad \Rightarrow \quad C_u \;=\; \frac{3}{3 - \lambda} \quad . \tag{62}$$

Hence, the solution is

$$u(x) \;=\; e^x + \frac{3\lambda\, x}{3 - \lambda} \tag{63}$$

for arbitrary $\lambda \neq 3$. This is elementary, since this is a system corresponding to a 1×1 matrix $\mathcal{K}$. Yet most inhomogeneous equations will not have kernels as simple as this one.

Example 8: Consider now the Fredholm equation

$$u(x) \;=\; \lambda \int_0^\pi \sin(x-t)\, u(t)\, dt \quad , \tag{64}$$

which is homogeneous. It does not admit solutions for general λ, and so has to be treated as an eigenvalue problem. Since $\sin(x-t) = \sin x \cos t - \cos x \sin t$, we set

$$\begin{aligned} C_u &= \int_0^\pi \cos t\, u(t)\, dt \\ S_u &= \int_0^\pi \sin t\, u(t)\, dt \quad , \end{aligned} \tag{65}$$

and the Fredholm equation becomes

$$u(x) \;=\; \lambda \Big\{ C_u \, \sin x - S_u \, \cos x \Big\} \quad . \tag{66}$$

This form can be fed directly into Eq. (65) and the integrals routinely evaluated. This yields two results:

$$\begin{aligned} S_u &= \lambda \left\{ C_u \int_0^\pi \sin^2 t\, dt - S_u \int_0^\pi \sin t \, \cos t\, dt \right\} \;=\; \frac{\pi}{2}\, \lambda\, C_u \\ C_u &= \lambda \left\{ C_u \int_0^\pi \sin t \, \cos t\, dt - S_u \int_0^\pi \cos^2 t\, dt \right\} \;=\; -\frac{\pi}{2}\, \lambda\, S_u \quad . \end{aligned} \tag{67}$$

Viable solutions are realized only for eigenvalues $\lambda = \pm 2i/\pi$, with eigenvector solutions

$$u(x) \;=\; -\frac{2}{\pi}\, C_u \, e^{\pm ix} \tag{68}$$

for arbitrary normalization constant C_u, which can be real or complex. Accordingly, the solution is a complex function.

• Observe that we have pursued a simultaneous equation protocol for securing the solution to this equation.

Example 9: For an illustration of matrix protocols, consider the homogeneous equation

$$\phi(x) \ = \ \lambda \int_{-1}^{1} (t+x)\,\phi(t)\,dt \quad . \tag{69}$$

The individual functions that the kernel comprises are

$$M_1(x) \ = \ 1\ , \quad M_2(x) \ = \ x \quad \text{and} \quad N_1(t) \ = \ t\ , \quad N_2(t) \ = \ 1 \quad . \tag{70}$$

It follows that

$$\mathcal{K} \ = \ \begin{pmatrix} 0 & 2/3 \\ 2 & 0 \end{pmatrix} \quad . \tag{71}$$

The secular equation and the eigenvalues are simply obtained

$$\begin{vmatrix} 1 & -2\lambda/3 \\ -2\lambda & 1 \end{vmatrix} \ = \ 1 - \frac{4\lambda^2}{3} \ = \ 0 \quad \Rightarrow \quad \lambda \ = \ \pm\frac{\sqrt{3}}{2} \quad . \tag{72}$$

The eigenvectors are quickly determined:

$$\begin{aligned} \phi_1(x) \ &= \ 1 + \sqrt{3}\,x \ , \quad \lambda \ = \ \frac{\sqrt{3}}{2} \\ \phi_2(x) \ &= \ 1 - \sqrt{3}\,x \ , \quad \lambda \ = \ -\frac{\sqrt{3}}{2} \quad , \end{aligned} \tag{73}$$

and they are of arbitrary normalization because the integral equation is homogeneous.