14. INTEGRAL EQUATIONS

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1 Overview

Integral equations are those where the function to be determined lies inside the integral, though perhaps also outside the integral also. They can often be an integrated version of a differential equation, with the added information of boundary conditions provided.

• There are two key types of integral equations that will serve as the focus of our studies: **A&W Sec. 16.1**

- if the limits of integration are fixed, it is called a **Fredholm equation**;
- if the limits are a variable, then one has a **Volterra equation**.

If the unknown function appears only under the integral sign, it is an integral equation of the *first kind*; otherwise it is of the second kind. Thus, a **Fredholm equation of the first kind** assumes the form

$$f(x) \;=\; \int_a^b K(x,\, t)\, \phi(t)\, dt \quad , \tag{1}$$

seeking a solution for $\phi(x)$ given known $f(x)$ and **kernel** $K(x,\, t)$, and

$$\phi(x) \;=\; f(x) + \lambda \int_a^b K(x,\, t)\, \phi(t)\, dt \tag{2}$$

is a **Fredholm equation of the second kind**.

A **Volterra equation of the first kind** assumes the form

$$f(x) \; = \; \int_a^x K(x,\, t)\, \phi(t)\, dt \quad , \tag{3}$$

seeking a solution for $\phi(x)$ given $f(x)$ and **kernel** $K(x,\, t)$, and

$$\phi(x) \; = \; f(x) + \lambda \int_a^x K(x,\, t)\, \phi(t)\, dt \tag{4}$$

is a **Volterra equation of the second kind**. For second kind cases, we retain explicit parametric λ presence as in certain cases it can serve as an eigenvalue. Both Fredholm and Volterra concepts can be extended to multiple dimensions. The boundaries a, b, x are influential in framing solutions.

• A second order differential equation can be recast as Volterra equation of the second kind. This can be a useful transformation as integrals are inherently stable in numerical applications, while differentiation can be imprecise and unstable to error propagation, for example in the Euler technique.

Consider the linear ODE

$$y'' + A(x)\, y' + B(x)\, y \; = \; \chi(x) \quad , \tag{5}$$

subject to boundary conditions

$$y(a) \; = \; y_0 \quad , \quad y'(a) \; = \; y_0' \quad . \tag{6}$$

Integrating the ODE on the interval $[a,\, t]$ yields

$$\begin{aligned} y'(t) \; &= \; -\int_a^t Ay'\, dX - \int_a^t By\, dX + \int_a^t \chi\, dX + y_0' \\ &= \; -A(t)y(t) + \int_a^t (A' - B)\, y\, dX + \int_a^t \chi\, dX + y_0' + A(a)\, y_0 \quad , \end{aligned} \tag{7}$$

where in deriving the second line, an integration by parts has been performed for the first term on the first line. Another integration over $a \leq t \leq x$ yields

$$\begin{aligned} y(x) \; = \; &-\int_a^x A(t)y(t)\, dt + \int_a^x dt \int_a^t \Big[A'(X) - B(X) \Big]\, y(X)\, dX \\ &+ \int_a^x dt \int_a^t \chi(X)\, dX + \Big[y_0' + A(a)\, y_0 \Big] (x - a) + y_0 \quad . \end{aligned} \tag{8}$$

The double integrals can be simplified to single integrals by reversing the orders of integrations and performing the simple one first. The result is

$$y(x) \;=\; f(x) + \int_a^x K(x,\,t)\,y(t)\,dt \tag{9}$$

for

$$\begin{aligned} f(x) \;&=\; \int_a^x (x-t)\chi(t)\,dt + \Big[y_0' + A(a)\,y_0\Big](x-a) + y_0 \\ K(x,\,t) \;&=\; (x-t)\Big[A'(t) - B(t)\Big] - A(t) \quad . \end{aligned} \tag{10}$$

This reformulation as an integral equation offers certain advantages to demonstrating existence and uniqueness of solution.

Example 1: Consider the linear oscillator equation and Cauchy boundary conditions:

$$y'' + \omega^2 y \;=\; 0 \quad \text{with} \quad y(0) \;=\; 0 \quad , \quad y'(0) \;=\; 1 \quad . \tag{11}$$

Thus we have $A(x) = 0$, $B(x) = \omega^2$ and $\chi(x) = 0$. It is then a simple matter to determine that the corresponding Volterra equation is

$$y(x) \;=\; x - \omega^2 \int_0^x (x-t)\,y(t)\,dt \quad . \tag{12}$$

While our ODE pedagogy leads us to quickly infer the solution $y(x) = [\sin\omega x]/\omega$, soon paths for obtaining this solution using the integral equation will become apparent.

Example 2: Now let us change the boundary conditions for the linear oscillator ODE:

$$y'' + \omega^2 y \;=\; 0 \quad \text{with} \quad y(0) \;=\; 0 \quad , \quad y(b) \;=\; 0 \quad . \tag{13}$$

The integration protocol leading to Eq. (10) is slightly modified, and the result is

$$y(x) \;=\; y_0' x - \omega^2 \int_0^x (x-t)\,y(t)\,dt \quad . \tag{14}$$

Herein, y_0' is an unknown, and it can be eliminated using the boundary condition at $x = b$. Thus,

$$y_0' \;=\; \frac{\omega^2}{b} \int_0^b (b-t)\, y(t)\, dt \quad . \tag{15}$$

With this, one can write down the integral equation in closed form. Yet it is desirable to clean it up. To do this, one divides the full interval of interest into two intervals $[0,\, x]$ and $[x,\, b]$ and uses the identity

$$\frac{x}{b}(b-t) - (x-t) \;=\; \frac{t}{b}(b-x) \tag{16}$$

Then the development leads to

$$\begin{aligned} y(x) \;&=\; -\omega^2 \int_0^x (x-t)\, y(t)\, dt + \omega^2\, \frac{x}{b} \int_0^b (b-t)\, y(t)\, dt \\ &=\; \omega^2 \int_0^x \frac{t}{b}\,(b-x)\, y(t)\, dt + \omega^2 \int_x^b \frac{x}{b}\,(b-t)\, y(t)\, dt \quad . \end{aligned} \tag{17}$$

Recognizing that the kernel of the integral can be formed as a single triangle function that is symmetric and continuous

$$K(x,\, t) \;=\; \begin{cases} \dfrac{t}{b}(b-x) \quad , & 0 \;\le\; t \;\le\; x \quad , \\[2ex] \dfrac{x}{b}(b-t) \quad , & x \;\le\; t \;\le\; b \quad , \end{cases} \tag{18}$$

one then arrives at the final form

$$y(x) \;=\; \omega^2 \int_0^b K(x,\, t)\, y(t)\, dt \quad , \tag{19}$$

a homogeneous Fredholm equation of the second kind. Accordingly, the type of the resultant integral equation formed from an ODE depends on the boundary conditions.

• In both these examples, observe that we now have a parameter $\lambda = \omega^2$ appearing, which may connect to eigenvalues of a multi-dimensional boundary value problem.

• Since the kernel is symmetric and continuous, it provides convolution of the function onto itself and is often referred to as the **Green's function** for this integral equation problem.

2 Transform Techniques

It is sometimes expedient to employ Laplace and Fourier transforms to facilitate solution, often via algebraic rearrangement before ultimate inversion. This is particularly useful when the integral equations involve functional convolutions. We illustrate by example.

A&W Sec. 16.2

Example 3: We solve here the generalized **Abel equation**

$$f(x) \;=\; \int_0^x \frac{\phi(t)}{(x-t)^\alpha}\, dt \quad , \quad 0 \;<\; \alpha \;<\; 1 \quad , \tag{20}$$

for $\phi(x)$ with $f(x)$ known. First take Laplace transforms of both sides, using the convolution theorem: if $\mathcal{L}_1 = \mathcal{L}(f_1)$ and $\mathcal{L}_2 = \mathcal{L}(f_2)$ for two functions f_1 and f_2, then

$$\mathcal{L}_1 \cdot \mathcal{L}_2 \;=\; \mathcal{L}\left\{ \int_0^t f_1(t-z)\, f_2(z)\, dz \right\} \quad . \tag{21}$$

Thus

$$\mathcal{L}(f) \;=\; \mathcal{L}(x^{-\alpha}) \cdot \mathcal{L}(\phi) \quad \Rightarrow \quad \mathcal{L}(\phi) \;=\; \frac{\mathcal{L}(f)}{\mathcal{L}(x^{-\alpha})} \;=\; \frac{s^{1-\alpha}}{\Gamma(1-\alpha)}\, \mathcal{L}(f) \; . \tag{22}$$

This can then be written

$$\frac{1}{s}\, \mathcal{L}(\phi) \;=\; \frac{\mathcal{L}(x^{\alpha-1}) \cdot \mathcal{L}(f)}{\Gamma(\alpha)\, \Gamma(1-\alpha)} \quad . \tag{23}$$

Then employing the reflection identity for the Gamma function, expressing the product of Laplace transforms on the right as a convolution, we arrive at

$$\int_0^x \phi(t)\, dt \;=\; \mathcal{L}^{-1}\left\{ \frac{1}{s}\, \mathcal{L}(\phi) \right\} \;=\; \frac{\sin \pi\alpha}{\pi} \int_0^x (x-t)^{\alpha-1}\, f(t)\, dt \quad . \tag{24}$$

Differentiation then derives the final solution

$$\phi(x) \;=\; \frac{\sin \pi\alpha}{\pi}\, \frac{d}{dx} \left\{ \int_0^x \frac{f(t)\, dt}{(x-t)^{1-\alpha}} \right\} \quad , \quad 0 \;<\; \alpha \;<\; 1 \quad . \tag{25}$$

This encompasses a fairly general set of integral functions $f(x)$, and so indicates how useful transform techniques can be.

• To assess the breadth of its applicability, consider the case of a power-law form $f(x) = x^n$. Then the solution integral can be computed analytically:

$$\phi(x) \ = \ \frac{\sin \pi\alpha}{\pi} \frac{d}{dx} \left\{ \int_0^x \frac{t^n \, dt}{(x-t)^{1-\alpha}} \right\} \ = \ \frac{\sin \pi\alpha}{\pi} \frac{d}{dx} \left\{ x^{n+\alpha} \int_0^1 \frac{u^n \, du}{(1-u)^{1-\alpha}} \right\} \ . \tag{26}$$

Using the Beta function, it follows that

$$\phi(x) \ = \ (n+\alpha) \frac{\sin \pi\alpha}{\pi} \, B(n+1, \, \alpha) \, x^{n+\alpha-1} \qquad . \tag{27}$$

Now, since the Abel equation is linear in both $f(x)$ and $\phi(t)$, one can simply for superpositions of power-laws for $f(x)$ and infer the solutions as simple sums. In other words, a Taylor series leads to a power series solution:

$$f(x) \ = \ \sum_{n=0}^{\infty} f_n x^n \quad \Rightarrow \quad \phi(x) \ = \ \sum_{n=0}^{\infty} \frac{f_n \, x^{n+\alpha-1}}{B(n+\alpha, \, 1-\alpha)} \qquad . \tag{28}$$

In developing this result, we have employed the reflection identity for the Gamma function to eliminate the sinusoidal factor.

Accordingly, a polynomial $f(x)$ will generate a solution $\phi(x)$ that is a polynomial of the same degree divided by the power law $x^{1-\alpha}$.

In addition, the exponential $f(x) = e^x$ generates the solution

$$f(x) \ = \ e^x \quad \Rightarrow \quad \phi(x) \ = \ \frac{x^{\alpha-1}}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+\alpha)} \qquad . \tag{29}$$

This can be extended to sinusoids and even inverse trigonometric functions, so that series solutions for a variety of inhomogeneous terms for the Abel equation are readily obtainable.

Example 4: Now consider the Fredholm equation

$$f(x) \;=\; \int_{-\infty}^{\infty} k(x-t)\,\phi(t)\,dt \quad . \tag{30}$$

Here $f(x)$ and $k(x)$ are known functions. The doubly-infinite range suggests the use of Fourier transforms. If we set $K(\omega) \equiv \mathcal{F}(k)$ and $\Phi(\omega) \equiv \mathcal{F}(\phi)$, then the Fourier convolution theorem gives

$$f(x) \;=\; \int_{-\infty}^{\infty} K(\omega)\,\Phi(\omega)\,e^{-i\omega x}\,d\omega \quad , \tag{31}$$

so that Fourier transformation yields

$$K(\omega)\,\Phi(\omega) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{i\omega x}\,dx \;\equiv\; \frac{F(\omega)}{\sqrt{2\pi}} \tag{32}$$

for $F(\omega) \equiv \mathcal{F}(f)$. Rearrangement then forms $\Phi(\omega)$ explicitly, and inversion determines

$$\phi(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(\omega)}{K(\omega)}\,e^{-i\omega x}\,d\omega \tag{33}$$

This again illustrates the facility of transform approaches when convolutions are present.