Example 3: Now consider Hermite's ODE in both its original and its putatively self-adjoint forms,

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) \ = \ 0 \quad , \tag{25}$$

which is not self-adjoint, and

$$\phi_n''(x) + [2n+1-x^2]\,\phi_n(x) \ = \ 0 \quad , \quad \phi_n(x) \ = \ e^{-x^2/2} H_n(x) \quad . \tag{26}$$

We have already established the orthogonality relation

$$\int_{-\infty}^{\infty} \phi_m(x)\,\phi_n(x)\,dx \ = \ 0 \quad , \quad m \ \neq \ n \quad . \tag{27}$$

The differential operator pertinent to Eq. (26) is

$$\mathcal{H}_x \ \equiv \ \frac{d^2}{dx^2} - x^2 \quad . \tag{28}$$

This applies to boundary conditions that $\phi_n \to 0$ as $|x| \to \infty$, with zero derivatives also there (guaranteed by the ODE itself); thus we have a Cauchy problem. The pertinent eigenvalue is $\lambda_n = -(2n+1)$ for eigenfunction ϕ_n.

Establishing the Hermitian character of $\mathcal{H}_x$ is straightforward using a protocol similar to that in Example 1. Let $L_x = d^2/dx^2 \equiv \ \mathcal{H}_x + x^2$. Then,

$$\begin{aligned} \int_{-\infty}^{\infty} y_1^*\, L_x\, y_2\, dx \ &= \ \cancel{\left[y_1^* \frac{dy_2}{dx} \right]_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} \frac{dy_1^*}{dx}\,\frac{dy_2}{dx}\,dx \\ &= \ \cancel{\left[-\frac{dy_1^*}{dx}\, y_2 \right]_{-\infty}^{\infty}} + \int_{-\infty}^{\infty} y_2\,\frac{d^2 y_1^*}{dx^2}\,dx \\ &= \ \left[\int_{-\infty}^{\infty} y_2^*\, L_x\, y_1\, dx \right]^* \quad . \end{aligned} \tag{29}$$

Then subtract the integral of $x^2 y_1^* y_2$ from both sides to yield

$$\int_{-\infty}^{\infty} y_1^*\, \mathcal{H}_x\, y_2\, dx \ = \ \left[\int_{-\infty}^{\infty} y_2^*\, \mathcal{H}_x\, y_1\, dx \right]^* \quad . \tag{30}$$

The self-adjoint nature of Eq. (26) on $(-\infty, \infty)$ is thus confirmed.

2 Inhomogeneous Cases: Green's Functions

We now proceed to identify the path for solution of inhomogeneous problems. We can return to our drumskin example, and drive the oscillations with a source term:

$$\mathcal{L}_r u + \mathcal{L}_\theta u + \mathcal{L}_t u \;=\; \underbrace{f(\vec{x})}_{\text{source}} \quad . \tag{31}$$

The task is to identify how to arrive at an analytic solution that can accommodate general forms for the driving term. To develop the formalism, we restrict considerations to Hermitian operators, solving

$$\mathcal{L}\, u(\vec{x}) - \lambda u(\vec{x}) \;=\; f(\vec{x}) \quad . \tag{32}$$

Presuming that we can solve the eigenvalue/eigenvector problem for $\mathcal{L}$, expand both $u(\vec{x})$ and $f(\vec{x})$ as series of eigenfunctions of $\mathcal{L}$:

$$u(\vec{x}) \;=\; \sum_n c_n\, u_n(\vec{x}) \quad , \quad f(\vec{x}) \;=\; \sum_n f_n\, u_n(\vec{x}) \quad . \tag{33}$$

Here $\mathcal{L}\, u_n(\vec{x}) = \lambda_n u_n(\vec{x})$. The differential equation in Eq. (32) then becomes

$$\sum_n c_n\, (\lambda_n - \lambda)\, u_n(\vec{x}) \;=\; \sum_n f_n\, u_n(\vec{x}) \quad . \tag{34}$$

Since the eigenvectors are linearly independent (and orthonormal),

$$c_n \;=\; \frac{f_n}{\lambda_n - \lambda} \quad . \tag{35}$$

To leverage the orthogonality property for eigenvectors of Hermitian operators, we form the inner product

$$u_m \cdot f \;\equiv\; \int_\Omega u_m^*(\vec{x})\, f(\vec{x})\, d^3x \;=\; \sum_n f_n\, u_m^* \cdot u_n \;=\; f_m \quad , \tag{36}$$

where we have used $u_m^* \cdot u_n = \delta_{mn}$. This result can be inserted into Eq. (35) and then into the series expansion for $u(\vec{x})$, yielding

$$\begin{aligned} u(\vec{x}) \;&=\; \sum_n \frac{u_n \cdot f}{\lambda_n - \lambda}\, u_n(\vec{x}) \\ &=\; \sum_n \frac{u_n(\vec{x})}{\lambda_n - \lambda} \int_\Omega u_n^*(\vec{x}\,')\, f(\vec{x}\,')\, d^3x' \quad . \end{aligned} \tag{37}$$

This can be expressed compactly as

$$u(\vec{x}) \;=\; \int_{\Omega} \mathcal{G}(\vec{x},\, \vec{x}\,') \, f(\vec{x}\,') \, d^3x' \quad , \tag{38}$$

where

$$\mathcal{G}(\vec{x},\, \vec{x}\,') \;=\; \sum_n \frac{u_n(\vec{x}) \, u_n^*(\vec{x}\,')}{\lambda_n - \lambda} \tag{39}$$

is called the **Green's function** of the linear differential operator $\mathcal{L}$. The significance of this function is that it describes how the source at position $\vec{x}\,'$ influences the field $u(\vec{x})$ at $\vec{x}$. This applies to both ODEs and PDEs.

• The Green's function formalism is appropriate to action at a distance forces like electrostatics and gravity – the Laplacian operator is Hermitian.

• It is also appropriate for scattering problems, where it is often called a **kernel**: the source term is then a potential modulation of an incoming or outgoing plane wave. This is the principle of the **Feynman propagator**.

• If the source is of infinite concentration, $f(\vec{x}) = \delta(\vec{x} - \vec{x}_0)$, then

$$u(\vec{x}) \;=\; \int_{\Omega} \mathcal{G}(\vec{x},\, \vec{x}\,') \, \delta(\vec{x}\,' - \vec{x}_0) \, d^3x' \;=\; \mathcal{G}(\vec{x},\, \vec{x}_0) \quad , \tag{40}$$

so that

$$\mathcal{L} \, \mathcal{G}(\vec{x},\, \vec{x}\,') - \lambda \, \mathcal{G}(\vec{x},\, \vec{x}\,') \;=\; \delta(\vec{x} - \vec{x}\,') \quad . \tag{41}$$

The Green's function satisfies the same differential equation, but with the source term replaced by the delta function. This is why the superposition of solutions mediated by the kernel of a linear operator defines the solution for general inhomogeneous source terms.

• **Important Nuance**: The exact mathematical nature of a Green's function *depends on both the linear operator, and the boundary conditions*: changing the boundary conditions will alter the form of $\mathcal{G}(\vec{x},\, \vec{x}\,')$.

• As is apparent in our previous eigenvalue/eigenvector exposition material, we can subsume the constant λ into the differential operator, in which case the eigenvalues change. As posed, the parameter λ can represent a separation constant for a multi-dimensional problem.

Example 4: To illustrate the formation of Green's functions in practice, we return to our familiar vibrating string problem, with $\mathcal{L} = d^2/dx^2$ and $\lambda = -k^2$. If it is pinned at its ends $x = 0, L$, then the operating ODE and boundary conditions are

$$\frac{d^2u}{dx^2} + k^2 u \ = \ f(x) \quad , \quad u(0) \ = \ 0 \ = \ u(L) \quad . \tag{42}$$

The operating volume is $\Omega \to [0,\, L]$, i.e., is one-dimensional. If this space is established via a separation of variables with the time proceeding first, then $k = \omega/c$ serves as a coupling constant. The eigenvalues for this ODE are

$$-\lambda \ \equiv \ k^2 \ \to \ -\lambda_n \ = \ \left(\frac{n\pi}{L}\right)^2 \quad . \tag{43}$$

The orthonormal eigenfunctions are

$$u_n(x) \ = \ \sqrt{\frac{2}{L}}\,\sin\frac{n\pi x}{L} \quad . \tag{44}$$

These are normalized on $[0,\, L]$, i.e.,

$$\int_0^L u_n^*(x)\, u_n(x)\, dx \ = \ 1 \quad . \tag{45}$$

The Green's function assumes the form

$$\mathcal{G}(\vec{x},\, \vec{x}\,') \ = \ \sum_n \frac{u_n(\vec{x})\, u_n^*(\vec{x}\,')}{\lambda_n - \lambda} \ = \ \frac{2}{L}\sum_n \frac{\sin(n\pi x/L)\,\sin(n\pi x'/L)}{k^2 - (n\pi/L)^2} \quad . \tag{46}$$

This then can be convolved with any $f(x')$ to derive the solution for any source term. Observe that much of the character of this Green's function resembles the Fourier series solution that one derived in earlier Chapters.

2.1 Inhomogenous Boundary Conditions

We now turn our attention to using Green's functions as a path to the solution of *homogeneous equations with inhomogeneous boundary conditions.* Consider the stretched string (again) and the wave equation with periodic spatial BCs:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \;=\; 0 \quad , \quad u(0,\,t) \;=\; 0 \;=\; u(L,\,t) \quad . \tag{47}$$

We consider two types of inhomogeneous Cauchy initial conditions:

1. $u_1(x,\,0) = u_0(x)$ and $\partial u_1(x,\,0)/\partial t = 0$, i.e., the initial displacement is specified, but the initial speed is zero;

2. $u_2(x,\,0) = 0$ and $\partial u_2(x,\,0)/\partial t = v_0(x)$, i.e., the initial speed is specified, but the string is initially in its rest position.

For both cases (with $u \to u_{1,2}$), we can accommodate the periodic spatial boundary conditions thus:

$$u(x,\,t) \;=\; \sum_{n=1}^{\infty} b_n(t)\,\sin\frac{n\pi x}{L} \quad , \tag{48}$$

an expansion in the spatial eigenfunctions. The derivatives in the wave equation become

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{L^2}\right) b_n(t)\,\sin\frac{n\pi x}{L} \\ \frac{\partial^2 u}{\partial t^2} &= \sum_{n=1}^{\infty}\frac{\partial^2 b_n}{\partial t^2}\,\sin\frac{n\pi x}{L} \quad . \end{aligned} \tag{49}$$

Insertion into the wave equation, combined with the orthogonality of the eigenfunctions, implies that we can isolate the time coefficients term-by-term:

$$\frac{\partial^2 b_n}{\partial t^2} \;=\; -\frac{n^2\pi^2 c^2}{L^2}\,b_n \quad . \tag{50}$$

This solves for $b_n(t)$ in sinusoidal form, and the general solution can be expressed thus:

$$u(x,\,t) \;=\; \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L} \quad , \tag{51}$$

• The boundary conditions for case 1 yield

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \;=\; u_0(x) \quad , \quad \sum_{n=1}^{\infty} \frac{n\pi c}{L}\, B_n \sin\frac{n\pi x}{L} \;=\; 0 \quad . \tag{52}$$

This yields the obvious solution $B_n = 0$, and the A_n coefficients are determined using the orthogonality relations for the trigonometric functions:

$$A_n \;=\; \frac{2}{L}\int_0^L u_0(x')\,\sin\frac{n\pi x'}{L}\,dx' \quad . \tag{53}$$

Assembling the pieces, one can write the final solution as ($u \to u_1$)

$$u_1(x,\,t) \;=\; \int_0^L g_1(x,\,x',\,t)\,u_0(x')\,dx' \quad , \tag{54}$$

when employing a Green's function

$$g_1(x,\,x',\,t) \;=\; \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\,\sin\frac{n\pi x'}{L}\,\cos\frac{n\pi ct}{L} \quad . \tag{55}$$

• In a similar analysis, the boundary conditions for case 2 yield

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \;=\; 0 \quad , \quad \sum_{n=1}^{\infty} \frac{n\pi c}{L}\, B_n \sin\frac{n\pi x}{L} \;=\; v_0(x) \quad . \tag{56}$$

This yields the obvious solution $A_n = 0$, and the B_n coefficients are determined using the orthogonality relations for the trigonometric functions:

$$B_n \;=\; \frac{2}{n\pi c}\int_0^L v_0(x')\,\sin\frac{n\pi x'}{L}\,dx' \quad . \tag{57}$$

Assembling the pieces, one can write the final solution as ($u \to u_2$)

$$u_2(x,\,t) \;=\; \int_0^L g_2(x,\,x',\,t)\,v_0(x')\,dx' \quad , \tag{58}$$

when employing a Green's function

$$g_2(x,\,x',\,t) \;=\; \frac{2}{\pi c}\sum_{n=1}^{\infty} \frac{1}{n}\,\sin\frac{n\pi x}{L}\,\sin\frac{n\pi x'}{L}\,\sin\frac{n\pi ct}{L} \quad . \tag{59}$$

• From these two cases, it is patently evident that altering the boundary conditions changes the Green's function. **Problem**: Illustrate g_1 and g_2 graphically in the $(x,\,x')$ plane for different t.