• An **inhomogeneous** form for the PDE results if a **force** or **source term** is added to the RHS of these equations.

∗ e.g. The inhomogeneous form of Laplace's equation is known as **Poisson's equation**.

• Observe that if $\psi \sim e^{\pm i\omega t}$, then the wave equation reduces to the Helmholtz equation with $k = \omega/c$, and if $\psi \sim e^{-\gamma t}$, then the diffusion equation reduces to the Helmholtz equation with $k^2 = \gamma/\kappa$.

• Even a homogeneous PDE can be complicated by the **boundary conditions** that enable its integration. The combined information is known as a **boundary value problem** (BVP). A homogeneous BVP requires not only a homogeneous PDE, but also homogeneous boundary conditions.

A unique solution requires sufficient boundary information but not too much, on a *boundary curve or surface*. The right level is determined by the type of equation. There are three standard types of boundary/initial conditions:

- **Dirichlet**: ψ is specified on the boundary;
- **Neumann**: $(\nabla\psi)_n$ is specified on the boundary;
- **Cauchy**: both ψ and $(\nabla\psi)_n$ are specified on the boundary.

Note that the subscript n denotes the direction *normal* to the boundary; this is required so as to propagate information into the region of interest.

Consider a 2D problem with a boundary curve given by $x = x(s),\ y = y(s)$, where s is a parameter describing the path length along the curve.

The tangent $\vec{t}$ and normal $\vec{n}$ vectors for the curve S are given by

$$\vec{t} = \left(\frac{dx}{ds}, \frac{dy}{ds}\right) \quad , \quad \vec{n} = \left(-\frac{dy}{ds}, \frac{dx}{ds}\right) \quad . \qquad (3)$$

Assume Cauchy conditions are given on S, i.e. $\psi(s)$ and $(\nabla\psi)_n$ are specified. Then along S,

$$\begin{aligned} \frac{d\psi}{ds} &= \frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds} \\ (\nabla\psi)_n &= -\frac{\partial\psi}{\partial x}\frac{dy}{ds} + \frac{\partial\psi}{\partial y}\frac{dx}{ds} \quad . \end{aligned} \tag{4}$$

Given the LHS quantities, we can solve for the two partial derivatives of ψ:

$$\begin{aligned} \frac{\partial\psi}{\partial x} &= -\frac{dy}{ds}(\nabla\psi)_n + \frac{dx}{ds}\frac{d\psi}{ds} \\ \frac{\partial\psi}{\partial y} &= \frac{dx}{ds}(\nabla\psi)_n + \frac{dy}{ds}\frac{d\psi}{ds} \quad . \end{aligned} \tag{5}$$

Observe that the LHS terms are multiplied by $(dx/ds)^2 + (dy/ds)^2 \equiv 1$. This then permits us to write the first two terms of a Taylor series expansion for $\psi(x, y)$ about some point along S, and extending off the curve to cover a small surrounding "patch" in (x, y).

To continue this Taylor series, we need three second-order derivatives, $\partial^2\psi/\partial x^2$, $\partial^2\psi/\partial x\,\partial y$ and $\partial^2\psi/\partial y^2$. If we differentiate Eq. (5) we get

$$\begin{aligned} \frac{d}{ds}\left(\frac{\partial\psi}{\partial x}\right) &= \frac{\partial^2\psi}{\partial x^2}\frac{dx}{ds} + \frac{\partial^2\psi}{\partial x\,\partial y}\frac{dy}{ds} \\ \frac{d}{ds}\left(\frac{\partial\psi}{\partial y}\right) &= \frac{\partial^2\psi}{\partial x\,\partial y}\frac{dx}{ds} + \frac{\partial^2\psi}{\partial y^2}\frac{dy}{ds} \quad . \end{aligned} \tag{6}$$

The left hand sides of these are known functions from the boundary conditions and the boundary curve. The third equation is provided by the PDE:

$$\mathcal{A}\frac{\partial^2\psi}{\partial x^2} + 2\mathcal{B}\frac{\partial^2\psi}{\partial x\,\partial y} + \mathcal{C}\frac{\partial^2\psi}{\partial y^2} = f\left(x,\, y,\, \psi,\, \frac{\partial\psi}{\partial x},\, \frac{\partial\psi}{\partial y}\right) \quad . \tag{7}$$

Accordingly, we have three equations for three unknown second derivatives. This can now be expressed as a matrix problem:

$$\begin{pmatrix} \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & \frac{dx}{ds} & \frac{dy}{ds} \\ \mathcal{A} & 2\mathcal{B} & \mathcal{C} \end{pmatrix} \begin{pmatrix} \frac{\partial^2\psi}{\partial x^2} \\ \frac{\partial^2\psi}{\partial x\,\partial y} \\ \frac{\partial^2\psi}{\partial y^2} \end{pmatrix} = \begin{pmatrix} \frac{d}{ds}\left(\frac{\partial\psi}{\partial x}\right) \\ \frac{d}{ds}\left(\frac{\partial\psi}{\partial y}\right) \\ f\left(x,\, y,\, \psi,\, \frac{\partial\psi}{\partial x},\, \frac{\partial\psi}{\partial y}\right) \end{pmatrix} \quad . \tag{8}$$

This can be solved unless the determinant of the matrix vanishes, i.e.,

$$\mathcal{A}\left(\frac{dy}{ds}\right)^2 - 2\mathcal{B}\,\frac{dx}{ds}\,\frac{dy}{ds} + \mathcal{C}\left(\frac{dx}{ds}\right)^2 \;=\; 0 \quad . \tag{9}$$

This **characteristic equation** specifies two *families* of curves known as **characteristics** of the original PDE in the $x - y$ plane, given by the ODE

$$\frac{dy}{dx} \;=\; \frac{1}{\mathcal{A}}\left(\mathcal{B} \pm \sqrt{\mathcal{B}^2 - \mathcal{A}\,\mathcal{C}}\right) \quad . \tag{10}$$

Hence, given the Cauchy conditions along a boundary curve, the Taylor series expansion for $\psi(x,\,y)$ can be *continued to second order* away from the boundary unless this curve is locally (or globally) parallel to a characteristic.

• By repeated differentiation, not shown here, this property can be extended to higher derivatives, with the same result!

• Thus, ψ can be extended away from the boundary curve, given Cauchy conditions on that curve, unless the curve is parallel to the characteristic: *the boundary information "propagates" along characteristics.*

∗ The two families of curves may or may not intersect, or even lie in the real plane. The nature of the characteristics is controlled by the sign of the **discriminant** $\mathcal{B}^2 - \mathcal{A}\,\mathcal{C}$.

There are three distinct cases for the characteristics that define constraints on the geometry of the boundary:

- $\mathcal{B}^2 - \mathcal{A}\,\mathcal{C} > 0 \;\Rightarrow$ **real** and intersecting characteristics. The PDE is said to be of the **hyperbolic** type. This case requires *the boundary to be open*, otherwise it will somewhere be parallel to a characteristic;
- $\mathcal{B}^2 - \mathcal{A}\,\mathcal{C} = 0 \;\Rightarrow$ **real** but parallel characteristics. The PDE is said to of the **parabolic** type, and also requires *an open boundary*;
- $\mathcal{B}^2 - \mathcal{A}\,\mathcal{C} < 0 \;\Rightarrow$ **complex** characteristics that lie outside the real plane. The PDE is said to of the **elliptic** type. A *closed boundary* is OK for these.

Example 1: To understand the significance of the characteristics, consider the 1D wave equation

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0 \quad , \tag{11}$$

which is actually a 2D PDE. It has $A = 1$, $B = 0$ and $C = -1/c^2$ so that the discriminant is positive, implying a *hyperbolic equation*. The characteristics are given by

$$\left(\frac{dt}{ds}\right)^2 - \frac{1}{c^2}\left(\frac{dx}{ds}\right)^2 = 0 \quad \Rightarrow \quad x = \text{const.} \pm ct \quad . \tag{12}$$

Now let us define two functions

$$\xi(x,\, t) = x - ct \quad , \quad \eta(x,\, t) = x + ct \quad , \tag{13}$$

that are constants along the respective characteristics.

Plot: Characteristics for the one-dimensional wave equation

We can change variables to these functions and thereby simplify the wave equation. First,

$$\frac{\partial \psi}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial \psi}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial \psi}{\partial \eta} = \frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial \eta} \quad . \tag{14}$$

From this we derive the second derivative

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial \xi}{\partial x} \cdot \frac{\partial}{\partial \xi}\left(\frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial \eta}\right) + \frac{\partial \eta}{\partial x} \cdot \frac{\partial}{\partial \eta}\left(\frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial \eta}\right) \tag{15}$$

so that

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} + 2\,\frac{\partial^2 \psi}{\partial \xi \partial \eta} \quad . \tag{16}$$

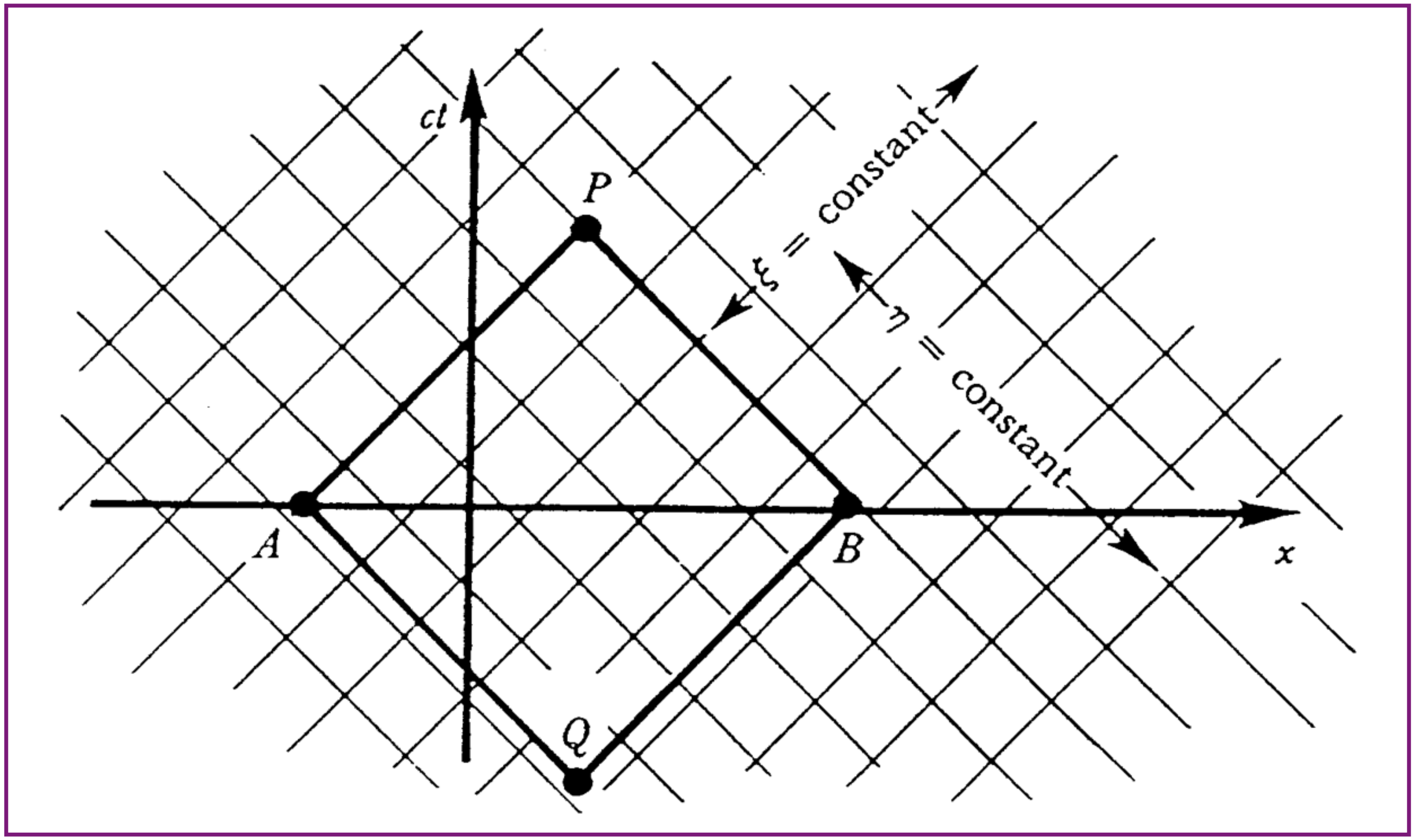
Likewise for the time derivative,

$$\frac{\partial \psi}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial \psi}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial \psi}{\partial \eta} = -c\left(\frac{\partial \psi}{\partial \xi} - \frac{\partial \psi}{\partial \eta}\right) \quad , \tag{17}$$

and the second derivative satisfies

$$\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} - 2\,\frac{\partial^2 \psi}{\partial \xi \partial \eta} \quad . \tag{18}$$

Characteristics for the 1D wave equation



- Characteristics for the one-dimensional wave equation. Being hyperbolic in type, these characteristics will be parallel to a closed boundary at select points, so an open boundary is needed for complete solution.

It is then evident that the **normal form** for the wave equation is

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \;=\; 0 \;=\; 4\,\frac{\partial^2 \psi}{\partial \xi \partial \eta} \quad . \tag{19}$$

It is then a trivial matter to define the solutions, as before:

$$\psi \;=\; f(\xi) + g(\eta) \quad , \tag{20}$$

and the characteristics are simply captured in this form.

• It thus becomes clear that information propagates along characterstics so that complete solution of a BVP is only attainable when the boundary is nowhere parallel to any characteristic.

All of the standard PDE examples discussed have discriminants with $B = 0$, so that their characteristic type is determined by the sign of $-AC$. Thus, one can quickly identify the types of PDEs and protocols for defining suitable boundary conditions.

Plot: Types of Common PDEs and Suitable Boundary Conditions

Characteristic Types of Common PDEs

All of the standard PDE examples discussed have discriminants with $B = 0$, so that their characteristic type is determined by the sign of $-AC$. Thus,

- **Laplace** equation: $A = 1 = C \;\Rightarrow\;$ *elliptic*
- **Wave** equation: $A = 1,\ C = -1/c^2 \;\Rightarrow\;$ *hyperbolic*
- **Diffusion** equation: $A = 1,\ C = 0 \;\Rightarrow\;$ *parabolic*
- **Helmholtz** & **Schrödinger** equations: $C = 0 \;\Rightarrow\;$ *parabolic*

Recipes for "safe" boundary conditions that guarantee solutions are:

- **Hyperbolic**: Cauchy conditions on an open boundary.
- **Parabolic**: Dirichlet *or* Neumann conditions on an open boundary.
- **Elliptic**: Dirichlet *or* Neumann conditions on a closed boundary.

2 Solution of Linear Second Order PDEs

There are four techniques that will be covered in the lectures, the last two being treated later in Chapter 13:

- Separation of variables
- Integral transforms
- Eigenfunction expansions
- Green's functions

These are not mutually exclusive, and may even be used in combination for a given problem. For example, it is sometimes possible to eliminate one of the independent variables (e.g. time) by separation of variables, and then apply one of the other techniques to the resulting (simpler) PDE.

2.1 Separation of Variables

The idea here is to reduce the PDE to a series of ODEs by assuming that the *dependence on the independent variables is separable*. Normally, the time variable is extracted first: **A&W Sec. 9.3**

$$\Psi(\vec{x}, t) \; = \; X(\vec{x})\, T(t) \quad , \tag{21}$$

where $\vec{x}$ is the position vector and t is the time variable. The spatial dependence $X(\vec{x})$ may then be further separated, depending on the geometry of the problem, as determined by the boundary conditions. For example:

- $X(\vec{x}) \; = \; X(x) \cdot Y(y) \cdot Z(z)$ (Cartesian coordinates)
- $X(\vec{x}) \; = \; R(r) \cdot \Theta(\theta) \cdot \Phi(\phi)$ (spherical coordinates)
- $X(\vec{x}) \; = \; R(r) \cdot \Phi(\phi) \cdot Z(z)$ (cylindrical coordinates)

All of the familiar examples of PDEs involving Laplacian operators are amenable to separation of the time variables, and the Laplacian takes on the form of the coordinate geometry.

Example 1: The wave equation. If we insert Eq. (21) into

$$\nabla^2 \Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} \;=\; 0 \quad , \tag{22}$$

the PDE reduces to

$$T\,\nabla^2 X - \frac{X}{c^2} \frac{\partial^2 T}{\partial t^2} \;=\; 0 \quad \Rightarrow \quad \frac{\nabla^2 X}{X} \;=\; \frac{1}{c^2} \frac{1}{T} \frac{\partial^2 T}{\partial t^2} \tag{23}$$

The LHS is a function of $\vec{x}$ only, while the RHS is only a function t. They must therefore both be equal to a constant, which is termed the **separation constant**, denoted $-k^2$ here (WLOG). Then

$$\frac{1}{T} \frac{\partial^2 T}{\partial t^2} \;=\; -k^2 c^2 \quad \Rightarrow \quad T \;\propto\; e^{\pm ikct} \tag{24}$$

describes the oscillatory time dependence. In principal, k could be imaginary, and this could define an exponential (growing or decaying) solution. The spatial solution is then

$$\nabla^2 X(\vec{x}) + k^2 X(\vec{x}) \;=\; 0 \quad , \tag{25}$$

which is the Helmholtz equation. Therefore we started with a hyperbolic equation and ended with a parabolic spatial equation.

Example 2: Diffusion equation. Again we try $\Psi(\vec{x}, t) = X(\vec{x})\, T(t)$ for

$$\nabla^2 \Psi - \frac{1}{\kappa} \frac{\partial \Psi}{\partial t} \;=\; 0 \quad . \tag{26}$$

The PDE reduces to

$$T\,\nabla^2 X - \frac{X}{\kappa} \frac{\partial T}{\partial t} \;=\; 0 \quad \Rightarrow \quad \frac{\nabla^2 X}{X} \;=\; \frac{1}{\kappa} \frac{1}{T} \frac{\partial T}{\partial t} \tag{27}$$

The LHS is again a function of $\vec{x}$ only, while the RHS is only a function t, so both can be set equal to a separation constant, $-k^2$. Then

$$\frac{1}{T} \frac{\partial T}{\partial t} \;=\; -k^2\,\kappa \quad \Rightarrow \quad T \;\propto\; e^{-k^2 \kappa t} \quad , \tag{28}$$

and the spatial solution is

$$\nabla^2 X(\vec{x}) + k^2 X(\vec{x}) \;=\; 0 \quad , \tag{29}$$

which is again the Helmholtz equation (parabolic → parabolic).

Both these examples will yield the Laplace equation $\nabla^2 X = 0$ if steady-state conditions are invoked. The spatial (Helmholtz or Laplace) equation can be further separated according to the boundary geometry. The three controlling Laplacian operators are defined by the coordinate system employed:

$$\begin{aligned}
\text{Cartesian:} \quad \nabla^2\psi &= \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \\
\text{Cylindrical:} \quad \nabla^2\psi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \\
\text{Spherical:} \quad \nabla^2\psi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \\
&\quad + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}
\end{aligned} \tag{30}$$

We now illustrate by example the character of the solution to the Helmholtz equation in these three popular coordinate systems.

Plot: Laplacian operators in common coordinate systems.

- Cartesian coordinates with $X(\vec{x}) = X(x)\cdot Y(y)\cdot Z(z)$:

$$X(\vec{x}) \sim \left\{e^{\pm i\alpha x}\right\}\cdot\left\{e^{\pm i\beta y}\right\}\cdot\left\{e^{\pm i\gamma z}\right\} \quad , \quad \alpha^2+\beta^2+\gamma^2 = k^2 \quad . \tag{31}$$

The notation $\{\}$ means any linear combination of terms of the indicated forms, subject to the constraint on the separation constants.

Further restrictions on α,β,γ are provided by the boundary conditions. For example, independence of z in a 2D geometry gives $\gamma = 0$, trivially. Periodic boundary conditions in x of period L given $\alpha = 2n\pi/L$ for integer n.

- Cylindrical coordinates with $X(\vec{x}) = R(r)\cdot\Phi(\phi)\cdot Z(z)$ yield

$$X(\vec{x}) \sim \left\{\begin{matrix} J_m(\sqrt{k^2+\alpha^2}\,\rho) \\ Y_m(\sqrt{k^2+\alpha^2}\,\rho)\end{matrix}\right\}\cdot\left\{e^{\pm im\phi}\right\}\cdot\left\{e^{\pm\alpha z}\right\} \quad , \tag{32}$$

where the J_m and Y_m are ordinary Bessel functions of the first and second kinds, respectively. We require m to be an integer for a single-valued, periodic solution in ϕ. In general, α is an arbitrary constant that can be complex. Periodic boundary conditions in z would set this up.

Laplacians in Common Coordinate Systems

$$\text{Cartesian:} \quad \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\text{Cylindrical:} \quad \nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\text{Spherical:} \quad \nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

- The coordinate system employed sets the form of the 3D Laplacian differential operator, which often then introduces the ODE for the special function appropriate to the spatial geometry of the PDE boundary value problem.

periodic solution in ϕ. In general, α is an arbitrary constant that can be complex. Periodic boundary conditions in z would set this up.

$*$ If we have "interior boundary conditions" specifying finite values at $\rho = 0$, no Y_m terms appear, since they diverge as $\rho \to 0$. Conversely, if we need finite values as $\rho \to \infty$, then we need to exclude J_m terms.

$*$ Often, we may have Dirichlet boundary conditions such as $X(\rho = a) = 0$, so that the values of k and α would be constrained to relate to the zeros of Bessel functions.

$*$ If $k = \alpha = 0$, the Bessel functions are replaced by $\rho^{\pm m}$.

• Spherical coordinates with $X(\vec{x}) = R(r) \cdot \Theta(\theta) \cdot \Phi(\phi)$ yield

$$X(\vec{x}) \sim \left\{ \begin{array}{c} j_\ell(kr) \\ n_\ell(kr) \end{array} \right\} \cdot \left\{ \begin{array}{c} P_\ell^m(\cos\theta) \\ Q_\ell^m(\cos\theta) \end{array} \right\} \cdot \left\{ e^{\pm im\phi} \right\} \quad , \tag{33}$$

where j_ℓ and n_ℓ are spherical Bessel functions, and P_ℓ^m and Q_ℓ^m are associated Legendre functions. We normally reject Q_ℓ^m on physical grounds as they diverge at $\cos\theta = \pm 1$. Again, m is an integer to guarantee a single-valued, periodic solution in ϕ.

$*$ If $k = 0$ and we have Laplace's equation, then we replace $j_\ell(kr)$ by r^ℓ and $n_\ell(kr)$ by $r^{-(\ell+1)}$.

• In summation, we note that the separation of variables technique generally gives the solution in the form of an infinite sum over one or more separation constants (e.g. multipole expansions in electrostatic or magnetostatic problems). This is most useful if the boundary/initial conditions can eliminate all but a few of the terms of the infinite sum(s).

We return to this technique shortly in the considerations of eigenfunction analysis, i.e. Sturm-Louiville theory.