2 Modified Bessel Functions $\mathbf{I}_\nu(\mathbf{x})$ and $\mathbf{K}_\nu(\mathbf{x})$

Wave equations in cylindrical coordinates, or the diffusion equation, often lead to the appearance of a modified Bessel differential equation for the cylindrical variable ρ: **A&W Sec. 11.5**

$$\rho^2 \frac{d^2y}{d\rho^2} + \rho \frac{dy}{d\rho} - (\rho^2 + \nu^2)\, y \;=\; 0 \quad . \tag{27}$$

The only modification is essentially to replace the independent variable $\rho \to iz$, so that by viewing the system in the complex plane, the solution is obviously a **modified Bessel function of the first kind**:

$$I_\nu(z) \;=\; e^{-\nu i\pi/2}\, J_\nu\Big(z\, e^{i\pi/2}\Big) \;\equiv\; \frac{1}{i^\nu}\, J_\nu(iz) \quad . \tag{28}$$

The normalization out the front is arbitrary, but is chosen to simplify the functional dependence near the origin $z = 0$. The Taylor series expansion is obtained by simple adaptation of Eq. (1), i.e.,

$$I_\nu(z) \;=\; \sum_{s=0}^{\infty} \frac{1}{s!\,(\nu+s)!} \left(\frac{z}{2}\right)^{\nu+2s} \quad . \tag{29}$$

The absence of the $(-1)^s$ factor in each term indicates that I_ν is not oscillatory in character, but rather exponential. For integer ν, we have

$$I_{-n}(z) \;=\; I_n(z) \quad . \tag{30}$$

The recurrence relations our routinely obtained from Eqs. (8) and (10) via the substitution $z \to iz$, yielding

$$\begin{aligned} I_{\nu-1}(z) - I_{\nu+1}(z) &\;=\; \frac{2\nu}{z}\, I_\nu(z) \quad , \\ I_{\nu-1}(z) + I_{\nu+1}(z) &\;=\; 2\, I_\nu'(z) \quad . \end{aligned} \tag{31}$$

From these, one can routinely demonstrate that $I_n(z)$ satisfies the ODE in Eq. (27), though this is guaranteed by the substitution protocol employed.

Similarly, minimal effort is required to obtain the generating function, employing $z \to iz$ and $t \to t/i$ in the Laurent series in Eq. (2):

$$g(z,\, t) \;\equiv\; \exp\left\{ \frac{z}{2} \left(t + \frac{1}{t} \right) \right\} \;=\; \sum_{n=-\infty}^{\infty} I_n(z)\, t^n \quad . \tag{32}$$

A **modified Bessel function of the second kind** or **MacDonald Function** can be defined to serve as the second solution to the modified Bessel ordinary differential equation:

$$K_\nu(z) \;=\; \frac{\pi}{2}\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin\nu\pi} \quad . \tag{33}$$

Observe that the $\cos\nu\pi$ factor does not appear due to the $n \to -n$ symmetry of $I_n(z)$. Such Bessel functions appear in the treatment of relativistic Maxwell Boltzmann distributions and synchrotron radiation theory. The recurrence relations are similar to those for the I_ν:

$$\begin{aligned} K_{\nu-1}(z) - K_{\nu+1}(z) \;&=\; -\frac{2\nu}{z}\,K_\nu(z) \quad , \\ K_{\nu-1}(z) + K_{\nu+1}(z) \;&=\; -2\,K'_\nu(z) \quad . \end{aligned} \tag{34}$$

The Wronskian for the two modified Bessel functions is also of an anticipated form:

$$I_\nu(z)K'_\nu(z) - I'_\nu(z)K_\nu(z) \;=\; -\frac{1}{z} \quad ; \tag{35}$$

the ODE modification does not alter the $1/z$ dependence of the Wronskian.

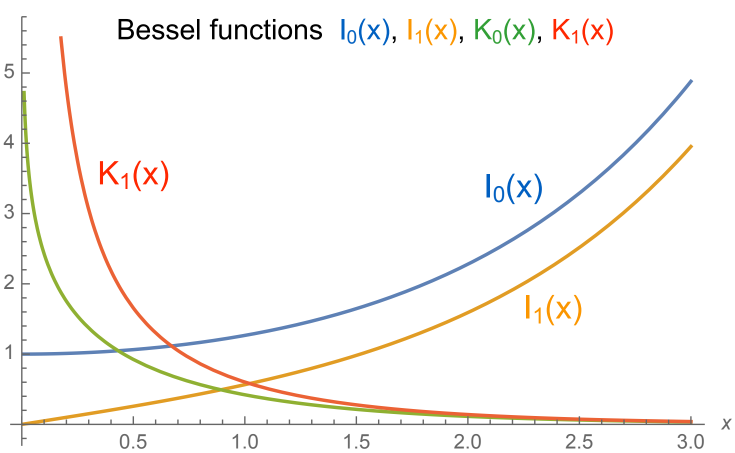

Figure 3: The modified Bessel functions $I_n(x)$ and $K_n(x)$ for $n = 0, 1$, illustrating the rising exponential character of the I_n and the declining exponential behavior of the K_n, which also diverge as $x \to 0$.

2.1 Integral Representations and Asymptotic Series

To obtain an integral representation analogous to the one derived for $J_n(z)$, we can insert $t \to ie^{i\theta}$ into the generating function in Eq. (32). Recognizing that the integral form for $J_n(z)$ is actually valid for arbitrary complex z, not just real arguments, permits the direct substitution $z \to iz$ in Eq. (32). Exploiting some symmetries in manipulating the integrals, the result is

$$I_n(z) \;=\; \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)\,d\theta \quad , \tag{36}$$

derived in A/W Ex. 11.5.14. Since the exponential portion of the integrand peaks strongly near $\theta \approx 0$ when $z \gg 1$, the method of steepest descents with $\cos\theta \approx 1 - \theta^2/2$ and $\cos n\theta \to 1$ then quickly yields an asymptotic form

$$I_n(z) \;\approx\; \frac{e^z}{\sqrt{2\pi z}} \quad , \quad z \gg 1 \quad , \tag{37}$$

which is independent of the index n. Higher order corrections are dependent on the index. A useful alternative integral representation is

$$I_\nu(z) \;=\; \frac{1}{\sqrt{\pi}\,\Gamma(\nu+1/2)}\left(\frac{z}{2}\right)^{\nu}\int_0^{\pi} e^{\pm z\cos\theta}\sin^{2\nu}\theta\,d\theta \quad . \tag{38}$$

- The most prominent integral representation for the modified Bessel function of the second kind is (see A/W Ex. 11.5.13)

$$K_\nu(z) \;=\; \int_0^{\infty} e^{-z\cosh\theta}\cosh(\nu\theta)\,d\theta \quad . \tag{39}$$

This also exhibits exponential character in z, and for $z \gg 1$, the dominant contribution to the integration is around $\theta \approx 0$, with $\cosh\theta \approx 1 + \theta^2/2$, quickly yielding a steepest descent result **A&W pp. 721**

$$K_\nu(z) \;\approx\; \sqrt{\frac{\pi}{2z}}\,e^{-z} \quad , \quad z \gg 1 \quad , \tag{40}$$

again independent of ν. Combining these asymptotic solutions, one observes consistency with the Wronskian in Eq. (35).

11. SPECIAL FUNCTIONS III: ORTHOGONAL POLYNOMIALS

Matthew Baring — Lecture Notes for PHYS 516, Fall 2022

1 Legendre Polynomials $P_n(x)$

A&W Sec. 12.1

Legendre functions emerge naturally in polar coordinate descriptions of physical quantities. For example, the Coulomb potential (inverse square force) for a charge q displaced from the origin at a distance $x = a$ can be cast in the form

$$\phi(r,\, \theta) \;=\; \frac{q}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} \quad . \tag{1}$$

This can be expressed as a series of Legendre polynomials using the generating function (for $z = \cos\theta$ and $t = \min(r/a,\, a/r) \leq 1$)

$$g(t,\, z) \;=\; \frac{1}{\sqrt{1 - 2zt + t^2}} \;=\; \sum_{n=0}^{\infty} P_n(z)\, t^n \quad , \quad |t| \;<\; 1 \quad . \tag{2}$$

A binomial series expansion of the generating function yields an infinite series in $2zt - t^2$, which can then be expressed as a double series in powers of t and z . Rearrangement of this yields finite series expressions for the Legendre polynomials

$$P_n(z) \;=\; \sum_{k=0}^{[n/2]} \frac{(-1)^k\, (2n-2k)!\, z^{n-2k}}{2^n\, k!\, (n-k)!\, (n-2k)!} \quad . \tag{3}$$

This is a result that could be generated from Frobenius' series solutions to Legendre's ODE, in the case of truncation for integer indices. From this, it is evident that P_n is odd or even according to the parity of n .

- It is simple to deduce Rodrigues' formula by forming the n^{th} derivative of x^{2n-2k} of each term in the summation, and invoking the binomial theorem:

$$P_n(x) \;=\; \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2^n\,k!\,(n-k)!}\left(\frac{d}{dx}\right)^n x^{2n-2k} \;=\; \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n \quad . \tag{4}$$

This form also clearly indicates that $P_n(x)$ is an n^{th} degree polynomial. Rodrigues' formula is useful in proving various properties of Legendre polynomials such as orthogonality. We can do this by successively integrating the appropriate inner product by parts. Without loss of generality, assume that $m < n$, so that

A&W Sec. 12.3

$$\begin{aligned}
\int_{-1}^{1} P_n(x)\,P_m(x)\,dx \;&\propto\; \int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n\;\frac{d^m}{dx^m}(x^2-1)^m\,dx\\
&=\; -\int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\;\frac{d^{m+1}}{dx^{m+1}}(x^2-1)^m\,dx\\
&\qquad\qquad\vdots\\
&=\; (-1)^m\int_{-1}^{1} \frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n\;\frac{d^{2m}}{dx^{2m}}(x^2-1)^m\,dx\\
&=\; 0 \quad .
\end{aligned} \tag{5}$$

At each integration by parts, the residual outside the integration contains an $(n-p)^{th}$ derivative of $(x^2-1)^n$, so it possesses a factor that is some non-zero power of x^2-1, which generates zero at the extremities $x = \pm 1$. The last integration contains a constant term from the derivative of $(x^2-1)^m$ (which is $(2m)!$), and so is simply evaluated as zero because $m < n$.

- The $m = n$ case can be used to establish the normalization:

$$2^{2n}\,(n!)^2\int_{-1}^{1}[P_n(x)]^2\,dx \;=\; \int_{-1}^{1}(1-x^2)^n\,(2n)!\,dx \;=\; \frac{(2n)!}{2}\,B\Big(\frac{1}{2},\,n+1\Big) \tag{6}$$

so that Legendre's duplication formula can then be applied to generate

$$\int_{-1}^{1}[P_n(x)]^2\,dx \;=\; \frac{2}{2n+1} \quad , \tag{7}$$

which defines the normalization of the Legendre polynomials.

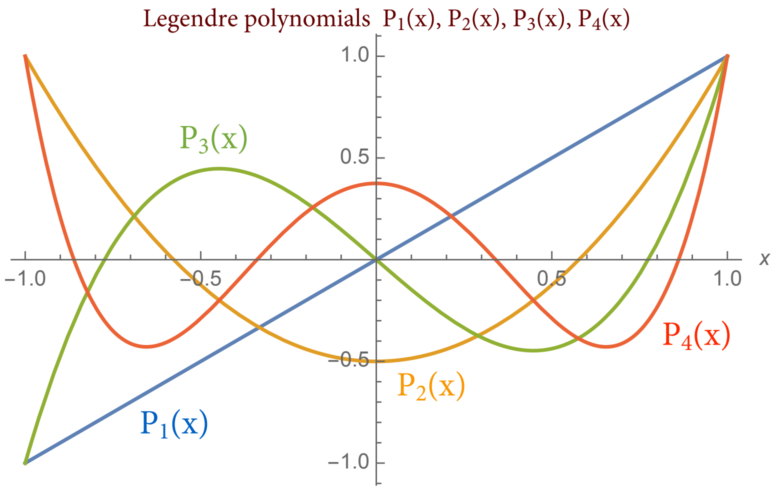

Figure 1: The Legendre polynomials $P_1(x)$, $P_2(x)$, $P_3(x)$ and $P_4(x)$, evaluated on the real axis, highlighting the parity being assigned by the value of index n, and the destructive interference that underpins their orthogonality.

- Observe that Rodrigues' form automatically implies the parity relation $P_n(-x) = (-1)^n \, P_n(x)$.

- Analytic continuation yields identical mathematical forms for the $P_n(z)$ in the domain $|z| > 1$.

- Differentiation of the generating function identity again leads to recurrence relations: **A&W Sec. 12.2**

$$\begin{aligned} (n+1)\, P_{n+1}(z) + nP_{n-1}(z) &= (2n+1) z \, P_n(z) \quad , \\ P'_{n+1}(z) - P'_{n-1}(z) &= (2n+1)\, P_n(z) \quad . \end{aligned} \tag{8}$$

For $|z| < 1$, these are actually stable to upward (or downward!) recurrence, and can be combined, differentiated and rearranged to yield Legendre's ODE

$$(1 - z^2)\, P''_n(z) - 2z\, P'_n(z) + n(n+1)\, P_n(z) \;=\; 0 \quad . \tag{9}$$

Such an equation arises naturally in the separation of variables technique for solution of PDEs in polar coordinates, with $z = \cos\theta$. Then n serves as a quantum number, often deriving from the restriction to single-valued solutions in the azimuthal coordinate ϕ.

1.1 Legendre Functions of the 2nd Kind $Q_n(x)$

The second solution to Legendre's ordinary differential equation constitutes a class of non-polynomial functions known as those of the second kind, $Q_n(z)$. By employing the Wronskian technique, for integer indices, **A&W Sec. 12.10**

$$Q_n(z) \;=\; P_n(z)\left\{\alpha_n + \beta_n \int^z \frac{dx}{(1-x^2)\,[P_n(x)]^2}\right\} \quad . \tag{10}$$

At the regular singular points $x = \pm 1$, since $P_n(x)$ is bounded and non-zero, the integral is logarithmic in character, with a residual functional dependence that can actually be expressed as a finite sum over $P_k(z)$ Legendre polynomials (G&R 8.831.2 and 8.831.3):

$$Q_n(z) \;=\; \frac{1}{2}\,P_n(z)\log_e\left|\frac{1+z}{1-z}\right| + \sum_{k=1}^{n}\frac{1}{k}P_{k-1}(z)\,P_{n-k}(z) \quad . \tag{11}$$

This form for $Q_n(z)$ is valid for all $|z| \neq 1$. Observe that the Q_n possess the opposite polarity of the P_n, namely $Q_n(-z) = (-1)^{n+1}Q_n(z)$; this can quickly be deduced from Eq. (11).

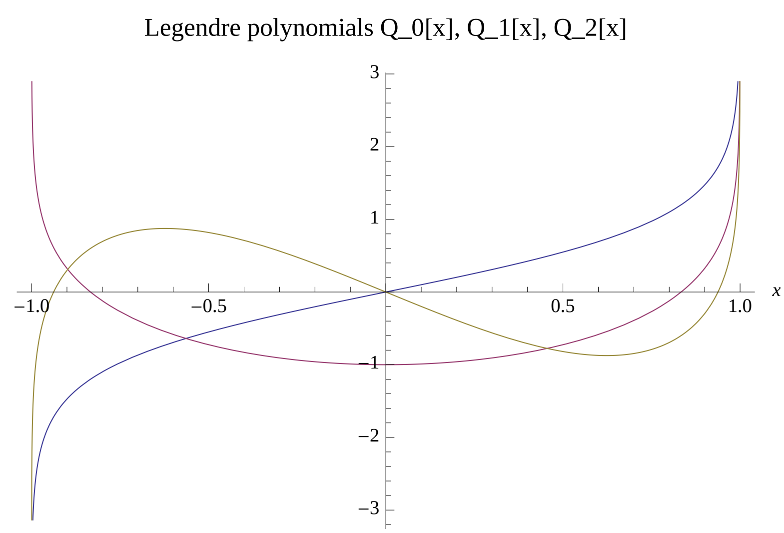

Figure 2: The Legendre polynomials $Q_0(x)$, $Q_1(x)$ and $Q_2(x)$, evaluated on the real axis, highlighting the parity assigned by the value of index n.

1.2 Associated Legendre Functions

As an extension to the Legendre functions we have considered so far, we have the **associated Legendre functions** $P_n^m(z)$, which for integer indices m and n are polynomials, which can be expressed in the Rodrigues' form **A&W Sec. 12.5**

$$P_n^m(z) \;=\; (-1)^m (1-z^2)^{m/2} \, \frac{d^m}{dz^m} P_n(z) \quad , \tag{12}$$

for $|m| \leq n$. They have a generating function

$$\frac{(2m)!}{2^m \, m! \, (1-2zt+t^2)^{m+1/2}} \;=\; \sum_{n=0}^{\infty} P_{n+m}^m(z) \, t^n \quad , \quad |t| \;<\; 1 \quad , \tag{13}$$

and satisfy the associated Legendre ODE

$$(1-z^2) \, \frac{d^2 y}{dz^2} - 2z \, \frac{dy}{dz} + \left[n(n+1) - \frac{m^2}{1-z^2} \right] y \;=\; 0 \quad . \tag{14}$$

Such functions capture the θ angular part of **spherical harmonics**, the angular solutions from the separation of variables technique for the Laplacian differential operator ∇^2. The m quantum number is attached, as usual, to the azimuthal dependence, which can only be single-valued, and the n quantum number is the constraint for truncated Frobenius' series solutions to the associated Legendre ODE.

- The $P_n^m(z)$ possess two sets of orthogonality relations:

$$\begin{aligned} \int_{-1}^{1} P_p^m(x) \, P_q^m(x) \, dx \;&=\; \frac{2\,\delta_{p,q}}{2q+1} \, \frac{(q+m)!}{(q-m)!} \quad , \\ \int_{-1}^{1} \frac{P_n^m(x) \, P_n^k(x)}{1-x^2} \, dx \;&=\; \frac{(n+m)!}{2m\,(n-m)!} \, \delta_{m,k} \quad , \quad 0 \;<\; m \;\leq\; k \;<\; n \quad . \end{aligned} \tag{15}$$

- The associated Legendre functions also possess an addition theorem that is germane to the multiplication of basis vectors connected by a spherical triangle. It follows from rotation group properties on the sphere, and is a useful tool for addition of various vector quantities such as angular momenta in quantum mechanics.