10. SPECIAL FUNCTIONS II: BESSEL FUNCTIONS

Matthew Baring — Lecture Notes for PHYS 516, Fall 2022

1 Bessel Functions $\mathbf{J}_\nu(\mathbf{x})$ and $\mathbf{N}_\nu(\mathbf{x})$

• Bessel functions naturally occur in problems with cylindrical symmetry, particularly for select differential operators such as the Laplacian ∇^2.

• For select problems, such as the Helmholtz PDE that involves the differential operator $\nabla^2 + k^2$, they occur in spherical polar coordinates too. In each case, the separation of variable technique employed in distilling solutions to the PDEs results in the Bessel ODE appearing for a select variable; this shall become apparent in the Chapter on PDEs.

1.1 Generating Function and Recurrence Relations

It is assumed, as a starting point, that the infinite series representation of the $J_n(z)$ Bessel function is its definition, namely **A&W Sec. 11.1**

$$J_n(z) \;=\; \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{z}{2}\right)^{n+2s} \quad , \quad n \;=\; 0,\, 1,\, 2, \dots \tag{1}$$

for arbitrary z in the complex plane, which is clearly oscillatory in nature. Consider the Laurent series of the two variable function

$$g(z,\, t) \;\equiv\; \exp\left\{ \frac{z}{2} \left(t - \frac{1}{t} \right) \right\} \;=\; \sum_{n=-\infty}^{\infty} y_n(z)\, t^n \quad . \tag{2}$$

To extract the coefficient of the t^n term on the LHS, we expand the exponential as a product of two Taylor series:

$$e^{zt/2} \cdot e^{-z/2t} \;=\; \sum_{r=0}^{\infty} \left(\frac{z}{2}\right)^r \frac{t^r}{r!} \;\; \sum_{s=0}^{\infty} (-1)^s \left(\frac{z}{2}\right)^s \frac{t^{-s}}{s!} \tag{3}$$

This double series can be re-labelled by setting $r = n + s$ for each s, with the restriction $n + s \geq 0$. The double series is then of the form

$$\sum_{n=-\infty}^{\infty} t^n \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{z}{2}\right)^{n+2s} \Theta(s+n) \quad . \tag{4}$$

Here $\Theta(x)$ is the Heaviside step function, and is zero for negative arguments. For $n \geq 0$, one then trivially assigns the coefficients of t^n to establish that $y_n(z) = J_n(z)$ in Eq. (1). Accordingly,

$$g(z,\, t) \;\equiv\; \exp\left\{ \frac{z}{2} \left(t - \frac{1}{t} \right) \right\} \;=\; \sum_{n=-\infty}^{\infty} J_n(z)\, t^n \tag{5}$$

is termed the **generating function** for ordinary Bessel functions $J_n(z)$.

For the $n < 0$ case, the double series is truncated at $s + n = 0$, and development appears to be more of a problem. If we proceed by using a substitution $t \to -1/t$ in the generating function, then since this still yields the same generating function, we have the result

$$g\Big(z,\, -\frac{1}{t}\Big) \;\equiv\; \exp\left\{ \frac{z}{2} \left(t - \frac{1}{t} \right) \right\} \;=\; \sum_{n=-\infty}^{\infty} J_{-n}(z)\, t^{-n} \quad , \tag{6}$$

again validated by the previous algebra for $n \geq 0$. A relabelling of $n \to -n$ then accesses the negative integers, and the complete result is proven.

• To compute Bessel functions of different indices n, it is often useful to employ the well-known **recurrence relations**, which are most easily derived from derivatives of the generating function. For example,

$$\sum_{n=-\infty}^{\infty} n\, J_n(z)\, t^{n-1} \;=\; \frac{\partial}{\partial t}\, g(z,\, t) \;=\; \frac{z}{2} \left(1 + \frac{1}{t^2} \right) \sum_{n=-\infty}^{\infty} J_n(z)\, t^n \quad . \tag{7}$$

By relabelling the summations so that all terms have the same power of t, one quickly arrives at the recurrence relation (applicable for non-integer n)

$$J_{n-1}(z) + J_{n+1}(z) \;=\; \frac{2n}{z}\, J_n(z) \quad . \tag{8}$$

This can be used to compute Bessel functions of high index, but such a process *is only stable for downward recurrence*, in the sense that rounding errors accumulate with upward recurrence.

- Another recurrence relation can be derived by differentiating with respect to z:

$$\sum_{n=-\infty}^{\infty} J_n'(z)\, t^n \;=\; \frac{\partial}{\partial z}\, g(z,\, t) \;=\; \frac{1}{2}\left(t - \frac{1}{t}\right) \sum_{n=-\infty}^{\infty} J_n(z)\, t^n \quad . \tag{9}$$

The same series index relabelling technique yields

$$J_{n-1}(z) - J_{n+1}(z) \;=\; 2\, J_n'(z) \quad . \tag{10}$$

This can routinely be differentiated, and combined with Eq. (8) to derive

$$x^2\, \frac{d^2 J_n}{dx^2} + x\, \frac{dJ_n}{dx} + (x^2 - n^2)\, J_n \;=\; 0 \quad , \tag{11}$$

i.e. Bessel's ODE. This is left as an exercise.

1.2 An Integral Representation

The next result that is derived from the generating function is the prominent integral representation for $J_n(z)$. For this, we set $t \to e^{i\theta}$ in Eq. (5). Separating the real and imaginary parts defines two Fourier series:

A&W pp. 679-80

$$\begin{aligned} \cos(z \sin\theta) \;&=\; J_0(z) + 2 \sum_{m=1}^{\infty} J_{2m}(z)\, \cos 2m\theta \quad , \\ \sin(z \sin\theta) \;&=\; 2 \sum_{m=0}^{\infty} J_{2m+1}(z)\, \sin(2m+1)\theta \quad . \end{aligned} \tag{12}$$

This then suggests the employment of orthogonality integrations, yielding

$$\begin{aligned}
\frac{1}{\pi}\int_0^{\pi} \cos(z\sin\theta)\,\cos n\theta\,d\theta &= J_n(z) \quad , \quad n = 0,\,2,\,4,\dots \\
\frac{1}{\pi}\int_0^{\pi} \sin(z\sin\theta)\,\sin n\theta\,d\theta &= J_n(z) \quad , \quad n = 1,\,3,\,5,\dots
\end{aligned} \tag{13}$$

These integrals are identically zero for choices of n (odd, even) alternate to those indicated. Adding these two together then yields

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\,d\theta \quad . \tag{14}$$

From this well-known integral form, the oscillatory character of J_n is evident.

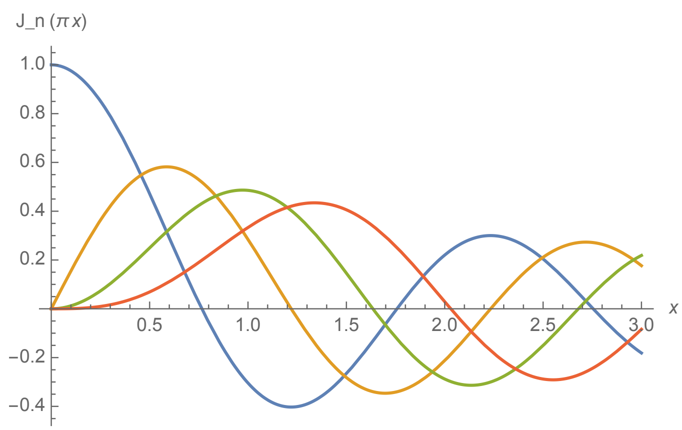

Figure 1: The Bessel functions $J_n(\pi x)$ for $n = 0, 1, 2, 3$ in trending from left to right, illustrating their oscillatory asymptotic character.

• To establish this more precisely, the integrand can be cast into complex exponential form, and the method of steepest descents used when $z \gg 1$. Then the arguments of the exponentials are $f(\theta) = \pm i\{z\sin\theta - n\theta\}$, so that zero derivative is realized when $\theta \approx \pi/2$, for large z. At this value, $f''(\theta) \approx \mp iz = \mp ze^{i\pi/2}$. It is then routine to obtain the asymptotic behavior

$$J_n(z) \approx \sqrt{\frac{2}{\pi z}}\cos\left\{z - \frac{\pi}{4}(2n+1)\right\} \quad , \quad z \gg 1 \quad . \tag{15}$$

This is clearly quasi-sinusoidal, and the phase offset scales as $\pi/4 + n\pi/2$, character that is clearly evinced in graphical depictions.

1.3 Neumann Functions and Wronskians

A&W Sec. 11.3

While the function $J_{-n}(z)$ is clearly proportional to $J_n(z)$ for integer n, it is routinely shown that $J_{-\nu}(z)$ is a linearly independent solution of Bessel's ODE from $J_\nu(z)$ if ν is not an integer. In fact it possesses a Laurent series and not a Taylor series. It is customary to define a Neumann function N_ν (often denoted by Y_ν) by the convenient linear combination

$$N_\nu(z) \;=\; \frac{\cos\nu\pi\, J_\nu(z) - J_{-\nu}(z)}{\sin\nu\pi} \tag{16}$$

as the second solution to Bessel's ODE. As $z \to 0$, the contribution from $J_{-\nu}(z)$ dominates, and the Laurent series can be used to demonstrate that

$$N_\nu(z) \;=\; -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{z}\right)^\nu + O(z^{1-\nu}) \tag{17}$$

Such a limit also applies for integer ν, where the mathematical form for $N_n(z)$ is given in G&R 8.403.2 as an infinite Frobenius series plus a term proportional to $J_n(z)\log_e(z/2)$. This is obtained by applying **l'Hopitâl's rule** to Eq. (16), i.e., setting $\nu = n + \delta$ and taking the limit $\delta \to 0$:

$$N_n(z) \;=\; \frac{1}{\pi}\left\{ \frac{\partial J_\nu(z)}{\partial\nu}\bigg|_{\nu=n} - (-1)^n\, \frac{\partial J_{-\nu}(z)}{\partial\nu}\bigg|_{\nu=n} \right\} \tag{18}$$

and employing the series expansion for J_n and J_{-n} for integer indices.

$*$ The Wronskian analysis (see below) could also be used to determine this.

• The Neumann functions obey the same recurrence relations as the J_n, a fact that can be simply reduced from the definition in Eq. (16). Accordingly,

$$\begin{aligned} N_{n-1}(z) + N_{n+1}(z) &= \frac{2n}{z}\, N_n(z) \quad , \\ N_{n-1}(z) - N_{n+1}(z) &= 2\, N_n'(z) \quad . \end{aligned} \tag{19}$$

These can be used to compute Neumann functions of various indices given that we know $N_0(z)$, which has a logarithmic divergence as $z \to 0$.

• One convenient protocol for computing $N_0(z)$ is to use the integral representation

$$N_\nu(z) \;=\; -\frac{2}{\pi}\int_0^\infty \cos\left(z\cosh t - \nu\pi/2\right)\cosh \nu t\, dt \quad , \quad |\nu| \;<\; 1 \quad . \tag{20}$$

This again suggests oscillatory behavior. There are integral representations for $N_n(z)$ that are analogous to ones for $J_n(z)$, but more complicated algebraically. In fact, the analog of Eq. (14) can be used to demonstrate the asymptotic result

$$N_n(z) \;\approx\; \sqrt{\frac{2}{\pi z}}\,\sin\left\{z - \frac{\pi}{4}\,(2n+1)\right\} \quad , \quad z \;\gg\; 1 \quad . \tag{21}$$

Hence, for $z \gg 1$, $N_n(z)$ and $J_n(z)$ are out of phase by $\pi/2$.

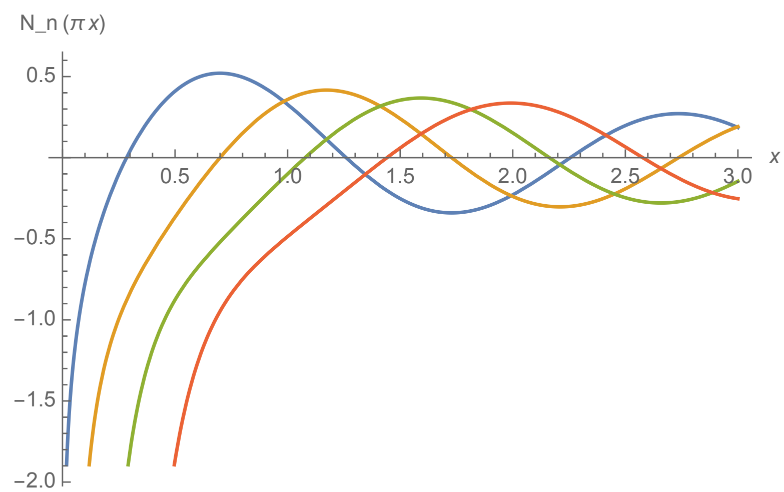

Figure 2: The Neumann functions $N_n(\pi x)$ for $n = 0, 1, 2, 3$ in trending from left to right, illustrating their oscillatory asymptotic character.

- Since the Bessel and Neumann functions satisfy Bessel's ODE

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)\, y \;=\; 0 \quad , \tag{22}$$

they satisfy the Wronskian, which for $p(x) = 1/x$ here is given by

$$W(x) \;\equiv\; y_1 y_2' - y_1' y_2 \;\propto\; \exp\left\{ -\int^x \frac{dt}{t} \right\} \;\propto\; \frac{1}{x} \quad . \tag{23}$$

One can then choose either $J_{-\nu}$ or N_ν to represent the second solution, given J_ν as the first. The constants of proportionality differ for the two choices, and can be determined by exploring the limiting forms for $z \to 0$, i.e. $J_\nu(z) \to (z/2)^\nu/\Gamma(\nu+1)$ and $N_\nu(z) \to -\Gamma(\nu)(z/2)^{-\nu}/\pi$, resulting in **A&W pp. 702-3**

$$\begin{aligned} J_\nu(z) J'_{-\nu}(z) - J'_\nu(z) J_{-\nu}(z) \;&=\; -\frac{2\sin\nu\pi}{\pi z} \quad , \\ J_\nu(z) N'_\nu(z) - J'_\nu(z) N_\nu(z) \;&=\; \frac{2}{\pi z} \quad . \end{aligned} \tag{24}$$

Remembering the technique for finding the second solution of a linear second order ODE from the first, we can further establish the identity

$$N_\nu(z) \;=\; N_\nu(x) - \frac{2}{\pi}\, J_\nu(z) \int_z^x \frac{dt}{t\,[J_\nu(t)]^2} \quad , \tag{25}$$

and an equivalent one for $J_{-\nu}$. These are principally of academic interest because they apply only to very limited ranges that do not capture any zeros of J_ν, since the integrals are divergent if they do.

∗ However, one niche use of this form is to ascertain the limiting behavior as $z \to 0$ of $N_\nu(z)$ given that we know that $J_\nu(z) \approx (z/2)^\nu/\Gamma(\nu+1)$ in this domain. When $z \ll x$, the $N_\nu(x)$ term on the RHS of Eq. (25) can be neglected. One then quickly arrives at Eq. (17), i.e.,

$$N_\nu(z) \;\approx\; -\frac{2}{\pi}\, \nu\Gamma(\nu)\,(2z)^\nu \int_z \frac{dt}{t^{2\nu+1}} \;=\; -\frac{\Gamma(\nu)}{\pi} \left(\frac{2}{z}\right)^\nu \quad . \tag{26}$$

In fact, J_ν and N_ν could be interchanged in the second solution formula, and the converse inference derived.

2 Modified Bessel Functions $\mathbf{I}_\nu(\mathbf{x})$ and $\mathbf{K}_\nu(\mathbf{x})$

Wave equations in cylindrical coordinates, or the diffusion equation, often lead to the appearance of a modified Bessel differential equation for the cylindrical variable ρ: **A&W Sec. 11.5**

$$\rho^2 \frac{d^2 y}{d\rho^2} + \rho \frac{dy}{d\rho} - (\rho^2 + \nu^2)\, y \;=\; 0 \quad . \tag{27}$$

The only modification is essentially to replace the independent variable $\rho \to iz$, so that by viewing the system in the complex plane, the solution is obviously a **modified Bessel function of the first kind**:

$$I_\nu(z) \;=\; e^{-\nu i\pi/2}\, J_\nu\bigl(z\, e^{i\pi/2}\bigr) \quad . \tag{28}$$

The normalization out the front is arbitrary, but is chosen to simplify the functional dependence near the origin $z = 0$. The Taylor series expansion is obtained by simple adaptation of Eq. (1), i.e.,

$$I_\nu(z) \;=\; \sum_{s=0}^{\infty} \frac{1}{s!\,(\nu+s)!} \left(\frac{z}{2}\right)^{\nu+2s} \quad . \tag{29}$$

The absence of the $(-1)^s$ factor in each term indicates that I_ν is not oscillatory in character, but rather exponential. For integer ν, we have

$$I_{-n}(z) \;=\; I_n(z) \quad . \tag{30}$$

The recurrence relations our routinely obtained from Eqs. (8) and (10) via the substitution $z \to iz$, yielding

$$\begin{aligned} I_{\nu-1}(z) - I_{\nu+1}(z) \;&=\; \frac{2\nu}{z}\, I_\nu(z) \quad , \\ I_{\nu-1}(z) + I_{\nu+1}(z) \;&=\; 2\, I'_\nu(z) \quad . \end{aligned} \tag{31}$$

From these, one can routinely demonstrate that $I_n(z)$ satisfies the ODE in Eq. (27), though this is guaranteed by the substitution protocol employed.

Similarly, minimal effort is required to obtain the generating function, employing $z \to iz$ and $t \to t/i$ in the Laurent series in Eq. (2):

$$g(z,\, t) \;\equiv\; \exp\left\{\frac{z}{2}\left(t + \frac{1}{t}\right)\right\} \;=\; \sum_{n=-\infty}^{\infty} I_n(z)\, t^n \quad . \tag{32}$$