## 1.3 Stirling's Series

It is impractical to compute the Gamma function for large arguments using either the limit form or the difference equation $\Gamma(z+1) = z\,\Gamma(z)$. Instead, we develop an approximation, due to Stirling, that has impressive precision, as we shall see. There are two paths to **Stirling's asymptotic series** for $\Gamma(z)$. First, the quick one, we use Euler's integral form for the Gamma function and computing it using the method of steepest descent:

**A&W Sec. 8.3**

$$\Gamma(z) \;=\; \int_0^\infty e^{-f(t,\,z)}\,dt \quad , \quad f(t,\,z) \;=\; t - (z-1)\log_e t \quad . \tag{22}$$

The argument of the exponential peaks at $\partial f/\partial t = 1 - (z-1)/t = 0$, i.e. when $t = z-1$, for which $f''(t,\,z) = 1/(z-1)$. It then quickly follows that

$$\Gamma(z) \;\approx\; \sqrt{2\pi(z-1)}\;\exp\Big\{-(z-1) + (z-1)\log_e(z-1)\Big\} \quad , \tag{23}$$

or

$$\log_e\Gamma(z) \;\approx\; \frac{1}{2}\log_e 2\pi + \left(z - \frac{1}{2}\right)\log_e z - z \quad , \tag{24}$$

where terms of order $1/z$ are neglected.

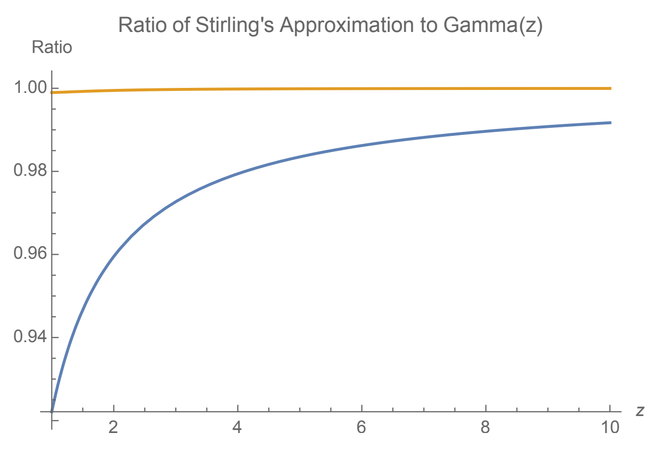


Figure 2: The ratios of the leading order Stirling approximation in Eq. (23) over $\Gamma(z)$ (lower curve), and that with the next order ($1/12z$) correction in Eq. (32) to $\Gamma(z)$ (upper curve), illustrating the precision of Stirling's series.

• The second is the protocol adopted by Arfken & Weber. We start with the **Euler-Maclaurin** formula for evaluating a definite integral:

$$\begin{aligned}\int_0^n f(x,\, z)\, dx \;&=\; \frac{1}{2}\, f(0,\, z) + f(1,\, z) + f(2,\, z) + \ldots + \frac{1}{2}\, f(n,\, z) \\ &\quad - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left[ f^{(2k-1)}(n,\, z) - f^{(2k-1)}(0,\, z) \right] \quad ,\end{aligned} \tag{25}$$

where the $f^{(j)}(x,\, z)$ are various $x$ derivatives of the (arbitrary) function $f(x,\, z)$. In this formula, the $B_{2k}$ are **Bernoulli numbers** from number theory, defined in Eq. (59) below, with $B_0 = 1$, $B_2 = 1/6$, $B_4 = -1/30$, etc, and $z$ is a parameter. This result is stated without proof, but is basically a refinement of the **trapezoidal rule** for integration including the series to define the remainder.

Now apply this result to $f(x,\, z) = 1/(z+x)^2$. Then

$$\begin{aligned}\frac{n}{z\,(z+n)} \;&=\; \frac{1}{2}\, f(0,\, z) + \sum_{k=1}^{n-1} \frac{1}{(z+k)^2} + \frac{1}{2}\, f(n,\, z) \\ &\quad - \sum_{k=1}^{\infty} (-1)^{2k-1}\, \frac{B_{2k}}{(2k)!} \left[ \frac{(2k)!}{(z+n)^{2k+1}} - \frac{(2k)!}{z^{2k+1}} \right] \quad ,\end{aligned} \tag{26}$$

Now we take the limit $n \to \infty$ of both sides. Re-labeling the second term on the first line of Eq. (26) via $k \to m+1$, we see that it approaches the polygamma function, $\psi^{(1)}(z+1)$, so that

$$\frac{1}{z} \;=\; \frac{1}{2z^2} + \psi^{(1)}(z+1) - \sum_{k=1}^{\infty} \frac{B_{2k}}{z^{2k+1}} \quad . \tag{27}$$

This can be integrated with respect to $z$ and rearranged thus:

$$\psi(z+1) \;\equiv\; \frac{d}{dz}\Big\{ \log_e\, z\Gamma(z) \Big\} \;=\; \mathcal{C}_1 + \log_e z + \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k\, z^{2k}} \quad . \tag{28}$$

To determine the constant of integration, we rearrange slightly, and integrate once more over a finite range arbitrarily close to infinity:

$$\lim_{\kappa\to\infty} \int_{\kappa}^{\kappa+1} \Big\{ \psi(z+1) - \log_e z \Big\}\, dz \;=\; \lim_{\kappa\to\infty} \left\{ \mathcal{C}_1 + \frac{1}{2}\, \log_e \frac{\kappa+1}{\kappa} + O\left(\frac{1}{\kappa}\right) \right\} \quad . \tag{29}$$

This then establishes that

$$
\begin{aligned}
\mathcal{C}_1 \;&=\; \lim_{\kappa\to\infty}\Big[\log_e z\Gamma(z) + z - z\log_e z\Big]_{\kappa}^{\kappa+1} \\
&=\; \lim_{\kappa\to\infty}\left[\log_e \frac{(\kappa+1)\Gamma(\kappa+1)}{\kappa\,\Gamma(\kappa)} + 1 - (\kappa+1)\log_e(\kappa+1) + \kappa\log_e\kappa\right] \qquad (30) \\
&=\; \lim_{\kappa\to\infty}\left[1 - \kappa\log_e\left(1+\frac{1}{\kappa}\right)\right] \;=\; 0
\end{aligned}
$$

Returning to the indefinite integral of Eq. (28), we have the asymptotic form

$$
\log_e\Gamma(z) \;=\; \mathcal{C}_2 + \left(z - \frac{1}{2}\right)\log_e z - z + \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)\,z^{2k-1}} \quad . \qquad (31)
$$

This constant of integration is determined by application of the Legendre doubling formula in Eq. (8):

$$
\log_e\Gamma(z+1/2) + \log_e\Gamma(z) \;=\; \log_e\Big\{2^{1-2z}\,\Gamma(2z)\,\sqrt{\pi}\Big\} \quad . \qquad (32)
$$

For each of the Gamma functions, insert Eq. (31), and then take the leading order contribution as $z \to \infty$. This results in the evaluation (a simple exercise) of $\mathcal{C}_2 = (\log_e 2\pi)/2$. The final result is **Stirling's asymptotic series** for the Gamma function $\Gamma(z)$:

$$
\log_e\Gamma(z) \;=\; \frac{1}{2}\log_e 2\pi + \left(z - \frac{1}{2}\right)\log_e z - z + \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)\,z^{2k-1}} \quad . \qquad (33)
$$

This is a precise form for computing the Gamma function when $z \gg 1$. Note that G&R 8.341.1 provides an integral or the series

$$
\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)\,z^{2k-1}} \;=\; \int_0^{\infty}\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{e^{-tz}}{t}\,dt \quad . \qquad (34)
$$

thereby providing an integral representation for $\log_e\Gamma(z)$.

$*$ Backing up a step and working with the derivative, one has the equivalent form for $\psi(z)$ obtainable directly from Eq. (28) with $\mathcal{C}_1 = 0$.

# 2 Functions Related to $\Gamma(z)$

## 2.1 Incomplete Gamma and Beta Functions

Generalizing the Euler integral definition of the Gamma function, we can define two **incomplete Gamma functions** valid in the right half of the complex plane: **A&W Sec. 8.5**

$$\gamma(z,\, x) \;=\; \int_0^x e^{-t}\, t^{z-1}\, dt \quad , \quad \Gamma(z,\, x) \;=\; \int_x^\infty e^{-t}\, t^{z-1}\, dt \quad . \tag{35}$$

It is clear that they satisfy the functional relationship

$$\gamma(z,\, x) + \Gamma(z,\, x) \;=\; \Gamma(z) \quad . \tag{36}$$

Observe that the error function is a special case:

$$\mathrm{erf}(x) \;=\; \frac{1}{\sqrt{\pi}}\, \gamma\Bigl(\frac{1}{2}\,,\, x^2\Bigr) \quad . \tag{37}$$

- There is also the **Beta function**, defined by **A&W Sec. 8.4**

$$B(p,\, q) \;=\; \frac{\Gamma(p)\, \Gamma(q)}{\Gamma(p+q)} \quad . \tag{38}$$

An integral form for it can be determined using that for $\Gamma(z)$:

$$\Gamma(p)\, \Gamma(q) \;=\; \int_0^\infty e^{-u}\, u^{p-1}\, du \int_0^\infty e^{-v}\, v^{q-1}\, dv \tag{39}$$

Now change variables via $u = x^2$ and $v = y^2$, and convert the resulting two-dimensional integral to polar coordinates via $x = r\cos\theta$, $y = r\sin\theta$ such that $dx\, dy = r\, dr\, d\theta$. The result is

$$\Gamma(p)\, \Gamma(q) \;=\; 4\int_0^\infty e^{-r^2}\, r^{2p+2q-1}\, dr \int_0^{\pi/2} \cos^{2p-1}\theta\, \sin^{2q-1}\theta\, d\theta \tag{40}$$

for the area mapped over a quarter plane. The radial integral is just a representation of another $\Gamma$ function, $\Gamma(p+q)/2$, so rearrangement yields

$$B(p,\, q) \;\equiv\; \frac{\Gamma(p)\, \Gamma(q)}{\Gamma(p+q)} \;=\; 2\int_0^{\pi/2} \cos^{2p-1}\theta\, \sin^{2q-1}\theta\, d\theta \quad . \tag{41}$$

Observe that the Beta function is symmetric in its arguments, i.e. under the interchange $p \leftrightarrow q$. Using the substitutions $\chi = \cos\theta$ and $t = \chi^2$ generates two alternative integral representations:

$$B(p,\, q) \;=\; \int_0^1 t^{p-1}\,(1-t)^{q-1}\,dt \;=\; 2\int_0^1 \chi^{2p-1}\,(1-\chi^2)^{q-1}\,d\chi \quad . \tag{42}$$

• Employing the Beta function, we can efficiently derive **Legendre's doubling formula** that we have used above. For $p = q = z$, we have

$$\frac{\Gamma(z)\,\Gamma(z)}{\Gamma(2z)} \;=\; \int_0^1 t^{z-1}(1-t)^{z-1}\,dt \;=\; 2^{2-2z}\int_0^1 (1-s^2)^{z-1}\,ds \quad , \tag{43}$$

where the substitution $t = (1+s)/2$ has been used, and then the even integrand used to restrict the integration to the range $[0,\, 1]$. The second integral is just half a Beta function ( $p \to 1/2$ , $q \to z$ ), so that

$$\frac{\Gamma(z)\,\Gamma(z)}{\Gamma(2z)} \;=\; 2^{1-2z}\frac{\Gamma(1/2)\,\Gamma(z)}{\Gamma(z+1/2)} \quad , \tag{44}$$

using the second form in Eq. (42). Simplifying and rearranging gives the **doubling formula** for the Gamma function:

$$\Gamma(2z) \;=\; \frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z)\,\Gamma(z+1/2) \quad . \tag{45}$$

There is also a tripling formula that can be found in G&R 8.335.2. Both are special cases of the **product theorem** of Gauss and Legendre for $\Gamma(z)$ :

$$\Gamma(nz) \;=\; \frac{n^{nz-1/2}}{(2\pi)^{(n-1)/2}}\prod_{k=0}^{n-1}\Gamma\Big(z+\frac{k}{n}\Big) \quad . \tag{46}$$

This can be proved using the limit form definition of $\Gamma(z)$ together with the infinite product for $\sin\pi z$ (Erdélyi, Vol I, p. 5). If one takes the derivative of the logarithm, this product theorem can be re-written as

$$\psi(nz) \;=\; \log_e n + \frac{1}{n}\sum_{k=0}^{n-1}\psi\Big(z+\frac{k}{n}\Big) \quad . \tag{47}$$

This result can be routinely proven using the integral identity in Eq. (20). Integration and exponentiation then yields the functional $z$-dependence of the product theorem for $\Gamma(nz)$ but not the multiplicative constant $\mathcal{C}_n$ .

## 2.2 Riemann Zeta Function

An important function in number theory that we have already used to test for convergence of series is the **Riemann zeta function** $\zeta(s)$, defined by

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} n^{-s} \quad , \quad s \;>\; 1 \quad . \tag{48}$$

When $s < 2$, the series is slow to converge, and so acceleration algorithms are required. One is the celebrated **Euler prime number product**, which is deduced by first forming

**A&W pp. 382-4**

$$\zeta(s)\,(1-2^{-s}) \;=\; 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \ldots\right) \quad , \tag{49}$$

thereby eliminating every second term of the series. Then, one forms

$$\begin{aligned} \zeta(s)\,(1-2^{-s})\,(1-3^{-s}) \;&=\; 1 + \frac{1}{3^s} + \frac{1}{5^s} + \ldots \\ &\quad - \left(\frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \ldots\right) \end{aligned} \tag{50}$$

so that now every third term, i.e. those involving multiples of $3^s$ in the denominators, is deleted. The process can be repeated for every prime number, with an overall remnant of just unity. Rearranging results in Euler's form:

$$\zeta(s) \;=\; \prod_{p\,=\,\mathrm{prime}}^{\infty} \frac{1}{1-p^{-s}} \quad . \tag{51}$$

For $s > 2$ this can be an efficient path to compute $\zeta(s)$. In `Mathematica` coding, convergence to $\zeta(1.5) = 2.61238$ is slow:

```
zetaprod[s_, n_]:= Product[ 1/(1-1/(Prime[k])^s), {k, 1, n} ]
  zetaprod[1.5, 10] = 2.4366
  zetaprod[1.5, 100] = 2.5845
  zetaprod[1.5, 1000] = 2.6069
  zetaprod[1.5, 10000] = 2.6111
```

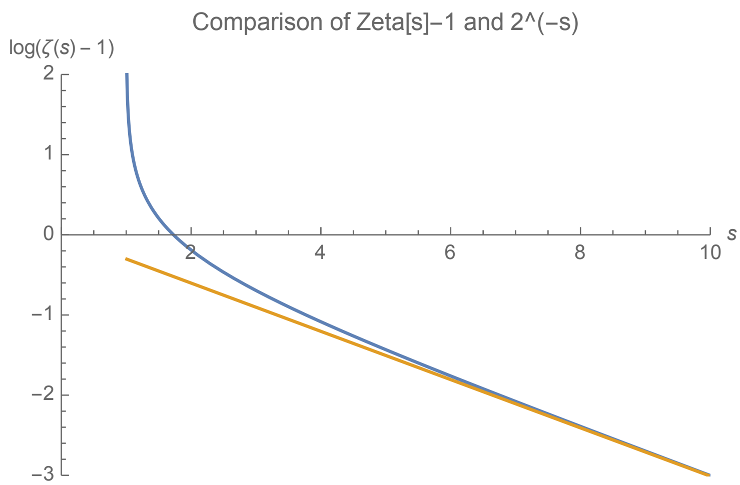


Figure 3: The comparison of the difference between the Riemann zeta function $\zeta(s)$ and unity, and the asymptotic tendency $2^{-s}$ that can be deduced from Euler's prime product formula.

• The product formula automatically implies that $\zeta(s) - 1$ asymptotically approaches $2^{-s}$ as $s \to \infty$. This is demonstrated graphically.

• More efficient paths for computing $\zeta(s)$ for $s < 2$ are afforded by Dirichlet series rearrangements such as

$$\eta(s) \;\equiv\; \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \;=\; \left(1 - 2^{1-s}\right) \zeta(s) \tag{51}$$

so that grouping $n = 2k - 1$ and $n = 2k$ terms together leads to

$$\zeta(s) \;=\; \frac{\eta(s)}{1 - 2^{1-s}} \;=\; \frac{1}{1 - 2^{1-s}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} \left\{ 1 - \left(1 - \frac{1}{2k}\right)^s \right\} \quad . \tag{52}$$

As $k$ becomes large, this series converges as $k^{-(1+s)}$, and so is reasonably efficient even right down to $s \approx 1$:

```
zetaetaser[s_, n_]:= Sum[ (2 k - 1)^(-s) (1 - (1 - 1/2/k)^ s),
                         {k, 1, n}]/(1 - 2^(1 - s))
  zetaetaser[1.5, 10] = 2.594
  zetaetaser[1.5, 30] = 2.60875
  zetaetaser[1.5, 100] = 2.61177
```

If one desires greater convergence speed for $\zeta(s)$, then one can manipulate using original (definitional) series for $\zeta(s+1)$. First, use Eq. (52) and form

$$\left\{1-2^{1-s}\right\}\zeta(s)-\frac{s\,\zeta(s+1)}{2^{s+1}} \;=\; \sum_{k=1}^{\infty}\left[\frac{1}{(2k-1)^s}-\frac{1}{(2k)^s}-\frac{s}{(2k)^{s+1}}\right] \quad , \tag{53}$$

where the series on the RHS now converges as $k^{-(2+s)}$. Now insert Eq. (52) evaluated for $s \to s+1$, so that its series also converges as $k^{-(2+s)}$. If we define

$$\mu(s) \;=\; \frac{s}{2(2^s-1)} \quad , \tag{54}$$

then rearranging the series identity for $\zeta(s)$ yields

$$\begin{aligned}\zeta(s) \;&=\; \lim_{n\to\infty}\zeta(s,\,n) \quad , \\ \zeta(s,\,n) \;&=\; \frac{1}{1-2^{1-s}}\sum_{k=1}^{n}\left[\frac{2k-1+\mu(s)}{(2k-1)^{s+1}}-\frac{2k+s+\mu(s)}{(2k)^{s+1}}\right] \quad .\end{aligned} \tag{55}$$

The precision of this accelerated series expansion is impressive: see Fig. 4.

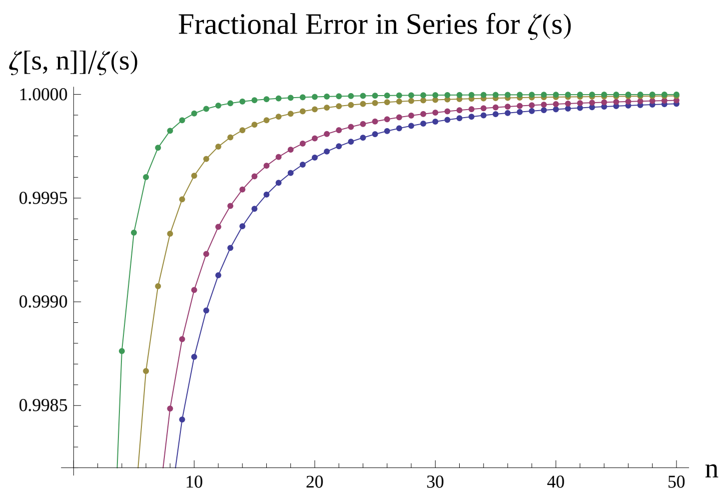


Figure 4: The ratio of the truncated series $\zeta(s,\,n)$ in Eq. (55) for the Riemann zeta function to $\zeta(s)$, itself, as a function of the number of terms $n$ summed. Cases are $s = 1.1, 1.2, 1.5, 2.0$ from bottom to top. Excellent precision is realized for all these $s$ choices by summing only 10 terms.

• It is also possible to define **generalized zeta functions**, namely via

$$\zeta(s,\, q) \;=\; \sum_{n=0}^{\infty} \frac{1}{(q+n)^s} \quad , \quad s \;>\; 1 \quad , \tag{56}$$

so that $\zeta(s,\, 1) \equiv \zeta(s)$. It is then quick to establish the integral representation

$$\zeta(s,\, q) \;=\; \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-qt}}{1 - e^{-t}}\, dt \tag{57}$$

by expressing the denominator of the integrand as a geometric series and then integrating term by term. Such integrals naturally emerge in Bose-Einstein statistics such as for the photon gas, i.e. the Planck spectrum.

∗ Observe that $\psi^{(m)}(z) = (-1)^{m+1} m!\, \zeta(m+1, z)$ establishes the relationship between polygamma functions and generalized zeta functions.

• It is interesting to establish the relationship between the Riemann zeta function and the rational fraction **Bernoulli numbers** $B_n$ of number theory. These are defined by the Taylor series expansion

$$\frac{x}{e^x - 1} \;=\; \sum_{n=0}^{\infty} \frac{B_n x^n}{n!} \quad . \tag{58}$$

The only non-zero Bernoulli number with odd index is $B_1 = -1/2$.

A host of trigonometric and hyperbolic functions, and derivatives and integrals possess series that involve Bernoulli numbers in their coefficients. For example, successively setting $x \to 2iz$ and $x \to -2iz$ in Eq. (58) and adding quickly leads to the series representation

$$\sum_{n=0}^{\infty} (-1)^n B_{2n} \frac{(2z)^{2n}}{(2n)!} \;=\; \frac{z}{\tan z} \;=\; 1 - 2 \sum_{n=1}^{\infty} \left( \frac{z}{\pi} \right)^{2n} \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \quad . \tag{59}$$

The second identity is obtained by recognizing that $\cot z = d/dz[\log_e(\sin z)]$ and using the infinite product representation for $\sin z$. It is then routine to establish the identity

$$\frac{B_{2n}}{(2n)!} \;=\; 2 \frac{(-1)^{n-1}}{(2\pi)^{2n}}\, \zeta(2n) \quad . \tag{60}$$

From this it follows that $\zeta(2n) = r_n \pi^{2n}$, where $r_n$ is a rational fraction. The series in Eq. (55) can be used to obtain rational approximations to $\pi^{2n}$.