9. SPECIAL FUNCTIONS I

Matthew Baring — Lecture Notes for PHYS 516, Fall 2022

1 The Gamma Function

The **Gamma function** or factorial function is ubiquitous in mathematics, appearing in a wide range of problems and techniques. It is unto its own, defining a class of special functions that is not directly related to others such as Bessel and hypergeometric functions, or orthogonal polynomials, which are focused upon below. Yet the Gamma function appears in so many of the results pertaining to these other classes of functions, as will become apparent.

• Notably, the Γ function is one of a general class of functions that do not satisfy any differential equation with rational coefficients; this sets it apart.

1.1 Definitions and Simple Properties

There are a variety of ways to define the Gamma function, and one can start with one and prove equivalence of other definitions. Here we start with **Euler's infinite limit** definition: **A&W Sec. 8.1**

$$\Gamma(z) \;\equiv\; \lim_{n\to\infty} \frac{1.2.3\ldots n}{z(z+1)(z+2)\ldots(z+n)}\, n^z \quad , \quad z \;\neq\; 0, -1, -2, -3, \ldots \tag{1}$$

This is applicable everywhere in the complex plane except at the simple poles $z = 0, -1, -2, -3, \ldots$ along the negative real axis. With this definition, it is trivial to determine that

$$\Gamma(1) \;\equiv\; \lim_{n\to\infty} \frac{1.2.3\ldots n}{1.2.3\ldots(n+1)}\, n \;=\; 1 \quad . \tag{2}$$

To identify the factorial character of $\Gamma(z)$, form

$$\begin{aligned}\Gamma(z+1) &= \lim_{n\to\infty} \frac{1.2.3\ldots n}{(z+1)(z+2)\ldots(z+n+1)}\, n^{z+1} \\ &= \lim_{n\to\infty} \frac{nz}{z+n+1}\, \frac{1.2.3\ldots n}{z(z+1)(z+2)\ldots(z+n)}\, n^{z}\end{aligned} \tag{3}$$

from which one quickly deduces the **difference equation**

$$\Gamma(z+1) \;=\; z\,\Gamma(z) \quad . \tag{4}$$

This takes the place of recurrence relations that emerge for other special functions such as orthogonal polynomials. Repeated application of this together with Eq. (2) yields

$$\Gamma(n+1) \;=\; n! \quad , \quad n \;=\; 0,1,2,\ldots \tag{5}$$

the factorial property of Γ for positive integer arguments.

- A second definition is **Euler's integral form**:

$$\Gamma(z) \;\equiv\; \int_0^\infty e^{-t}\, t^{z-1}\, dt \quad , \quad \mathrm{Re}(z) \;>\; 0 \quad . \tag{6}$$

The restriction on z to the right-half plane is needed to guarantee convergence of the integral. Using the change of variables $t = u^2$, this can be related to a generalization of Gauss' error integral:

$$\Gamma(z) \;=\; 2\int_0^\infty e^{-u^2}\, u^{2z-1}\, du \quad , \quad \mathrm{Re}(z) \;>\; 0 \quad . \tag{7}$$

From the Chapter on integration, the value of this integral has been already computed for $z = 1/2$, yielding $\Gamma(1/2) = \sqrt{\pi}$. From this, for all integer n, we determine that $\Gamma(n+1/2) \propto \sqrt{\pi}$, via the factorial property:

$$\begin{aligned}\Gamma\Big(n+\frac{1}{2}\Big) &= \Big(n-\frac{1}{2}\Big)\Big(n-\frac{3}{2}\Big)\ldots\frac{1}{2}\,\Gamma\Big(\frac{1}{2}\Big) \\ &= \frac{(2n-1)(2n-2)(2n-3)(2n-4)\ldots 3.2.1}{2^n\,(2n-2)(2n-4)\ldots 2}\,\sqrt{\pi} \\ &= 2^{1-2n}\,\frac{\Gamma(2n)}{\Gamma(n)}\,\sqrt{\pi} \quad , \quad n \;=\; 1,2,3,\ldots \quad ,\end{aligned} \tag{8}$$

which is a recasting of the **doubling formula** (G&R 8.335.1).

To prove the equivalence of Euler's integral and limit forms, consider the integral

$$F(z,\, n) \;=\; \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt \quad , \tag{9}$$

for z with positive real part. This tends to Γ as $n \to \infty$ because

$$\lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n \;=\; e^{-t} \quad \Rightarrow \quad \lim_{n\to\infty} F(z,\, n) \;=\; \Gamma(z) \quad . \tag{10}$$

The well-known limit tending to e^{-t} is easily proven by taking logarithms and expanding the L.H.S. as a Taylor series. Now, changing variables in Eq. (9) to $u = t/n$, the result can be integrated by parts n times to yield

$$\begin{aligned} \frac{F(z,\, n)}{n^z} \;&=\; \int_0^1 (1-u)^n\, u^{z-1}\, du \;=\; \frac{n}{z} \int_0^1 (1-u)^{n-1}\, u^z\, du \\ \ldots \;&=\; \frac{n(n-1)\ldots 2.1}{z(z+1)(z+2)\ldots(z+n)} \quad , \end{aligned} \tag{11}$$

a result that can be proved by mathematical induction. It is then trivial to establish equivalence of Eq. (6) with Euler's limit definition in Eq. (1).

- A third definition that has its uses is **Weierstrass' product form**:

$$\frac{1}{\Gamma(z)} \;=\; z\, e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} \quad \text{where} \quad \gamma \;=\; 0.577215665\ldots \tag{12}$$

is the famous **Euler-Mascheroni constant**. The equivalence of this to the Euler definition is quickly shown. Take the inverse of the **Euler limit**:

$$\begin{aligned} \frac{1}{z\,\Gamma(z)} \;&=\; \lim_{k\to\infty} \frac{1}{k^z} \prod_{n=1}^{k} \frac{n+z}{n} \;=\; \lim_{k\to\infty} e^{-z \log_e k} \prod_{n=1}^{k} \left(1 + \frac{z}{n}\right) \\ &=\; \lim_{k\to\infty} \left\{ e^{-z \log_e k} \prod_{n=1}^{k} e^{z/n} \right\} \times \lim_{k\to\infty} \left\{ \prod_{n=1}^{k} \left(1 + \frac{z}{n}\right) e^{-z/n} \right\} \quad . \end{aligned} \tag{13}$$

The logarithm of the first factor in Eq. (13) is γz, using the definition

$$\lim_{k\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} - \log_e k\right) \;\equiv\; \gamma \quad , \tag{14}$$

which describes the logarithmic divergence of the harmonic series, and Weierstrass' form in Eq. (12) quickly follows.

• This new tool then facilitates the derivation of the **reflection identity** for the $\Gamma(z)$, a result that maps the left and right halves of the complex plane onto each other:

$$\Gamma(z)\,\Gamma(1-z) \;=\; \frac{\pi}{\sin \pi z} \quad . \tag{15}$$

This is a powerful tool for effecting convenient manipulations of the Γ function, in particular for analytically continuing results from Euler's integral representation. We prove it by forming

$$\frac{1}{\Gamma(z)}\,\frac{1}{\Gamma(-z)} \;=\; z\,(-z)\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\left(1-\frac{z}{n}\right) \;=\; -z^2\,\frac{\sin \pi z}{\pi z} \tag{16}$$

using the infinite product formula for the sine function (G&R 1.431.1), not proven here, and then setting $\Gamma(1-z) = -z\,\Gamma(-z)$ and rearranging.

• From the reflection formula we can immediately deduce that $\Gamma(z)$ has no zeros in the complex plane, and an infinite number of simple poles at integer values along the negative real axis, and at $z = 0$.

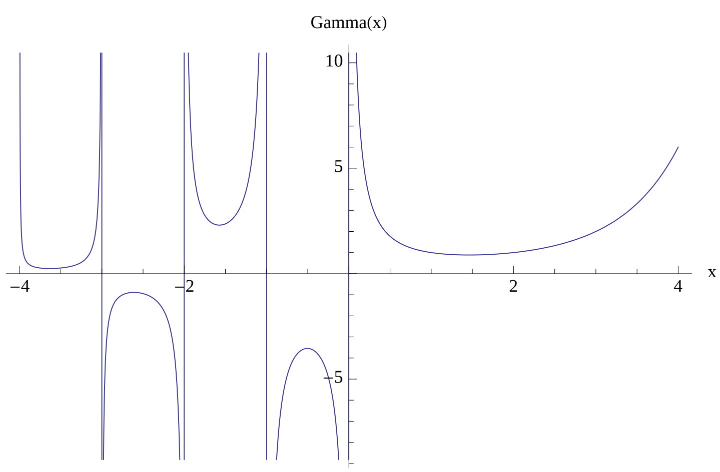

Figure 1: The Gamma function $\Gamma(x)$ evaluated on the real axis, highlighting the singularities at zero and negative integers.

1.2 Polygamma Functions

The first set of functions related to $\Gamma(z)$ that broaden our repertoire are the **polygamma functions**, that are sequential derivatives of the logarithm of the Gamma function. The first of these is the **digamma function** $\psi(z)$, defined via **A&W Sec. 8.2**

$$\psi(z) \equiv \frac{d}{dz}\log_e \Gamma(z) = -\frac{d}{dz}\left\{\log_e z + \gamma z + \sum_{n=1}^{\infty}\left[\log_e\left(1+\frac{z}{n}\right) - \frac{z}{n}\right]\right\} \quad (17)$$

and employing Weierstrass' product form for purposes of expediency. Arfken & Weber work from the limit definition of $\Gamma(z)$ and define polygamma functions in terms of the factorial. Here we adopt the more standard definitions appearing in the literature. The derivative in Eq. (17) is routine:

$$\psi(z) + \gamma + \frac{1}{z} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right) = \sum_{n=1}^{\infty}\frac{z}{n(n+z)} \quad . \quad (18)$$

For modest z, this series form defines an efficient protocol for computation, particularly when deploying an accelerated convergence algorithm; for large z, the Stirling series in Eq. (28) below is a preferred method.

- From the definition of $\psi(z)$ it is almost trivial to derive the functional relationship

$$\psi(z+1) = \psi(z) + \frac{1}{z} \quad . \quad (19)$$

A useful integral form is from G&R 8.361.7:

$$\psi(z) + \gamma = \int_0^1 \frac{t^{z-1}-1}{t-1}\, dt \quad , \quad (20)$$

which is routinely computed for $0.5 \lesssim z \lesssim 5$.

- Sequential derivatives lead to the definition of **polygamma functions**:

$$\psi^{(m)}(z) = \frac{d^{m+1}}{dz^{m+1}}\left\{\log_e \Gamma(z)\right\} = (-1)^{m+1} m! \sum_{n=0}^{\infty}\frac{1}{(z+n)^{m+1}} \quad , \quad (21)$$

for $m = 1, 2, 3, \ldots$, generally a rapidly convergent series. Contrast this with the factorial definition of Arfken & Weber that removes the $n = 0$ term.

1.3 Stirling's Series

It is impractical to compute the Gamma function for large arguments using either the limit form or the difference equation $\Gamma(z+1) = z\,\Gamma(z)$. Instead, we develop an approximation, due to Stirling, that has impressive precision, as we shall see. There are two paths to **Stirling's asymptotic series** for $\Gamma(z)$. First, the quick one, we use Euler's integral form for the Gamma function and computing it using the method of steepest descent: **A&W Sec. 8.3**

$$\Gamma(z) \;=\; \int_0^\infty e^{-f(t,\,z)}\,dt \quad , \quad f(t,\,z) \;=\; t-(z-1)\log_e t \quad . \tag{22}$$

The argument of the exponential peaks at $\partial f/\partial t = 1-(z-1)/t = 0$, i.e. when $t = z-1$, for which $f''(t,\,z) = 1/(z-1)$. It then quickly follows that

$$\Gamma(z) \;\approx\; \sqrt{2\pi(z-1)}\;\exp\Big\{-(z-1)+(z-1)\log_e(z-1)\Big\} \quad , \tag{23}$$

or

$$\log_e \Gamma(z) \;\approx\; \frac{1}{2}\log_e 2\pi + \left(z-\frac{1}{2}\right)\log_e z - z \quad , \tag{24}$$

where terms of order $1/z$ are neglected.

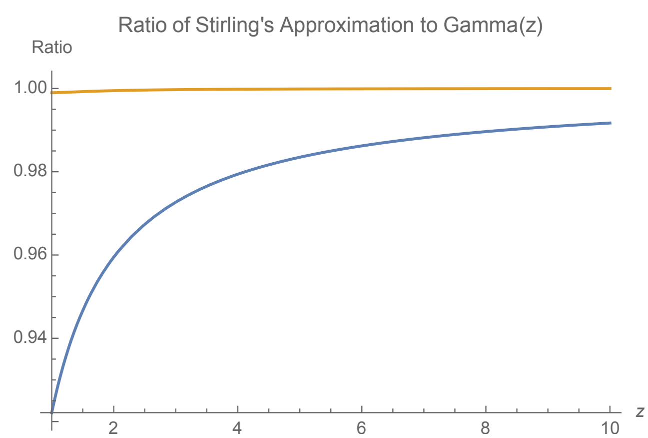

Figure 2: The ratios of the leading order Stirling approximation in Eq. (23) over $\Gamma(z)$ (lower curve), and that with the next order ($1/12z$) correction in Eq. (32) to $\Gamma(z)$ (upper curve), illustrating the precision of Stirling's series.