

7 Tensors

We now extend this algebra to define a **4-tensor of rank 2** to be a set of 16 quantities $T^{\mu\nu}$, which under coordinate transformations map like a product of two 4-vectors (which are 4-tensors of rank one). One can similarly define tensors of higher rank, i.e. with more indices. A rank 2 tensor looks like

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Sec. 2.6

$$\begin{bmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{bmatrix} \cdot [B^0 \ B^1 \ B^2 \ B^3] \quad (58)$$

This expands to a 4×4 matrix, and this will be the natural representation that will be adopted below for tensors suited to the theories of electrodynamics, fluid dynamics and general relativity.

There are three alternative forms describing a second-rank 4-tensor that extend the two forms we have encountered for 4-vectors:

$$\begin{aligned} T^{\mu\nu} &: \text{contravariant} \\ T^\mu_\nu \text{ or } T_\nu^\mu &: \text{mixed} \\ T_{\mu\nu} &: \text{covariant} \end{aligned} \quad (59)$$

The connection between the different types of components is determined by the general rule: *raising or lowering a space index (1, 2, 3) changes the sign of the component, while raising or lowering the time index (0) does not.* So,

$$T_{00} = T^{00} \quad , \quad T_{01} = -T^{01} \quad , \quad T_{11} = T^{11} \quad , \quad \text{etc.} \quad (60)$$

Technically, there are two types of mixed components. However, in practice, these are often identical due to the diagonality of the $\eta_{\alpha\beta}$ Minkowski metric tensor (matrix), and so in such cases we represent them without detailed attention to which index is raised or lowered. We have the relations

$$T_0^0 = T^{00} \quad , \quad T_0^1 = -T^{01} \quad , \quad T_1^0 = -T^{01} \quad , \quad T_1^1 = -T^{11} \quad , \quad \text{etc.} \quad (61)$$

A tensor is **symmetric** if $T^{\mu\nu} = T^{\nu\mu}$, or is **antisymmetric** if $T^{\mu\nu} = -T^{\nu\mu}$. Obviously, the diagonal elements of an antisymmetric tensor are all zero.

* For a symmetric tensor, $T^\mu_\nu = T_\nu^\mu$, so we just write it T^μ_ν .

- In every tensor equation (e.g. Maxwell's equations in electrodynamics), all terms must contain identical and identically-placed (i.e., raised or lowered) free indices, as opposed to *dummy indices* that are summed over. Free indices can be shifted up or down to derive alternative forms of the tensor equation, but such operations must be executed simultaneously to all terms.

The property that tensors map like products of 4-vectors under Lorentz transformations immediately identifies the transformation relations for a 4-tensor in boosting between two inertial frames K and K' :

$$T_{\mu\nu} = \frac{\partial x_\mu}{\partial x'_\alpha} T'_{\alpha\beta} \frac{\partial x_\nu}{\partial x'_\beta} \equiv \Lambda_\mu^\alpha T'_{\alpha\beta} \Lambda_\nu^\beta \quad , \quad (62)$$

where the primes denote evaluation of quantities in the K' frame.

* Observe that the combined pre- and post-multiplication protocol by Lorentz boost matrices is needed in order to preserve proper manipulations of tensor operations on vectors. This is immediately deducible by constructing the tensors as products of column and row vectors.

* Observe also that the boost transformations are also rank 2 tensors, and the placement of the indices, covariant in both numerator and denominator, yields contravariant α and β indices in the numerator. This preserves the signs of the space elements to the Λ_μ^α that are present in Eq. (50).

- One can form scalars from tensors, just as we did for 4-vectors, by summing over the indices. This reduces the number of free indices in an operation called **index contraction**, an example of which is

$$T \equiv T^\mu_\mu = \sum_{\mu=0}^3 T^\mu_\mu \quad , \quad (63)$$

which forms the **trace** of the tensor matrix. In general, contracting any pair of indices reduces the rank of a tensor by 2.

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Sec. 2.7

7.1 Special Tensors

Two special tensors are now introduced. The first is the **unit 4-tensor** δ_ν^μ ,

$$\delta_\nu^\mu = \begin{cases} 1 & , \text{ if } \mu = \nu & , \\ 0 & , \text{ if } \mu \neq \nu & . \end{cases} \quad (64)$$

This evinces the property that for any four-vector A^ν ,

$$\delta_\nu^\mu A^\nu = A^\mu \quad . \quad (65)$$

One can then raise or lower the indices of this tensor to obtain the **metric tensor** ($g^{\mu\nu}$ in contravariant form, $g_{\mu\nu}$ in covariant form). In flat spacetime, the metric tensor assumes the familiar **Minkowski metric** form

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Sec. 2.10

$$g^{\mu\nu} = g_{\mu\nu} \equiv \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta_{\mu\nu} \quad . \quad (66)$$

This now serves as the index raising and lowering operator for 4-vectors:

$$\begin{aligned} g_{\alpha\beta} A^\beta &= A_\alpha \quad , \quad g^{\alpha\beta} A_\beta = A^\alpha \\ A^\nu A_\nu &= g_{\alpha\beta} A^\alpha A^\beta = g^{\alpha\beta} A_\alpha A_\beta \quad . \end{aligned} \quad (67)$$

- A key special property of these tensors is that their form is the same in all coordinate systems, for example Cartesian and spherical polar configurations. Yet, specifically, they are invariant under Lorentz transformations:

$$g_{\mu\nu} = \Lambda_\mu^\alpha g_{\alpha\beta} \Lambda_\nu^\beta \quad . \quad (68)$$

This is easily confirmed using Eq. (50) for the flat spacetime case $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$.

- This invariance suggests that metric tensors are tied to the spacetime geometry. Consider now the space four-vector $A^\nu \rightarrow dx^\nu$. Then we have

$$ds^2 = dx^\nu dx_\nu = g_{\alpha\beta} dx^\alpha dx^\beta \quad , \quad (69)$$

which is the differential form of Eq. (41) for flat spacetime. In a gravitational field, $g_{\mu\nu}$ no longer assumes this simple form, and can in fact be non-diagonal, for example capturing the frame-dragging arising in the rotating Kerr metric.

- Another useful special tensor is the **completely antisymmetric unit tensor of 4th rank**, $\epsilon^{\alpha\beta\gamma\delta}$. This is the tensor whose components change sign under the interchange of any two indices. This antisymmetry constraint renders all components with two identical indices necessarily zero. The normalization property sets its nonzero components to be ± 1 .

- * Yet we still have a freedom of overall sign, so we set $\epsilon^{0123} = 1$.

- * There are a total of 24 non-vanishing components out of the 256.

There is also an anti-symmetric tensor of 3rd rank $\epsilon^{\alpha\beta\gamma}$, that like its 4th rank counterpart, is also known as the **Levi-Civita symbol**.

7.2 Anti-symmetric Tensors

- An anti-symmetric tensor has 12 nonzero components, and the antisymmetry constraint yields only six independent elements. Therefore, we can write it in the form

$$T^{\mu\nu} = \begin{pmatrix} 0 & p_x & p_y & p_z \\ -p_x & 0 & -a_z & a_y \\ -p_y & a_z & 0 & -a_x \\ -p_z & -a_y & a_x & 0 \end{pmatrix}, \quad (70)$$

without loss of generality. This implies that the tensor can be considered as being constructed *using two purely spatial vectors*, \mathbf{p} and \mathbf{a} . We represent this as two sets, one for each of the covariant and contravariant forms:

$$T^{\mu\nu} = (\mathbf{p}, \mathbf{a}) \quad , \quad T_{\mu\nu} = (-\mathbf{p}, \mathbf{a}) \quad . \quad (71)$$

If we perform a reflection of all the spatial coordinates, then the components with a single time index (0) switch sign, but those with two spatial indices (i.e. no temporal ones) do not. This is tantamount to the vector \mathbf{p} *reversing directions* under reflections (*odd parity*); it is then called a **polar vector** (hence the symbol).

- * Examples of polar vectors in physics include linear momentum \mathbf{p} and the electric field \mathbf{E} (e.g. in a dipole configuration).

On the other hand, the \mathbf{a} vector does not change signs under such a reflection operation (*even parity*). If it were a linear superposition of two polar vectors, it would then itself be polar. Instead, what if it is the cross product of two vectors: $\mathbf{a} = \mathbf{b} \times \mathbf{c}$? If both \mathbf{b} and \mathbf{c} are both polar, then \mathbf{a} does not change sign under this inversion, reproducing its even parity. It is then termed an **axial vector** (hence the symbol).

Plot: Polar and axial vectors under reflections

* Angular velocity vectors $\boldsymbol{\omega} \times \mathbf{r}$ describing rotations, orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ are examples of axial vectors.¹

The cross product serves as a convenient representation of axial vectors, since then the purely spatial elements of the tensor assume a form $a_k = b_i c_j - b_j c_i$ for cyclic permutation of the indices $i, j, k \rightarrow x, y, z$. This then re-distributes the spatial indices in the positions that exactly match the a_k elements given in $T^{\mu\nu}$ above. We emphasize that this is not a unique choice, but *it is a convenient one for electromagnetism*, for which cross product structure in the form of curls appears in the field equations.

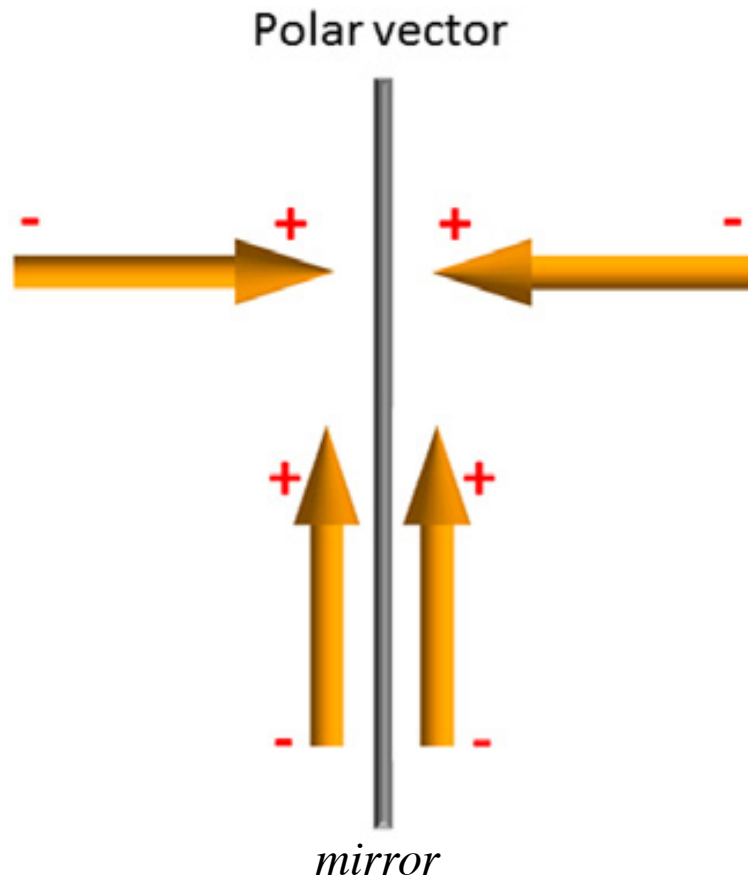
* Thus, we expect that the fundamental tensor for electromagnetism will be anti-symmetric, with the electric field components occupying the \mathbf{p} elements and the magnetic field components occupying the \mathbf{a} elements.

• Any tensor $T_{\mu\nu}$ can be decomposed into the sum of its symmetric, $(T_{\mu\nu} + T_{\nu\mu})/2$, and anti-symmetric, $(T_{\mu\nu} - T_{\nu\mu})/2$ parts. These parts can often constitute different physical content within one umbrella description, an example being the matter and electromagnetic components of the relativistic **energy-momentum tensor**.

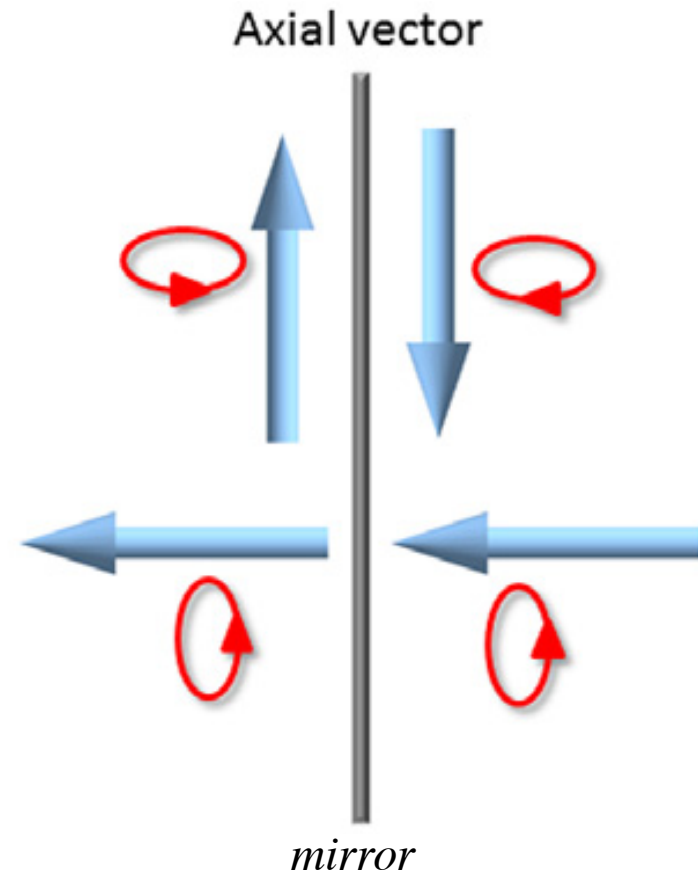
¹See the Feynman Lectures at http://www.feynmanlectures.caltech.edu/I_52.html

Polar and Axial Vectors

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Electric dipole



Magnetic dipole

7.3 Differentiation and Integration

As a last segment of this tensor summary, we touch upon differentiation and integration. It is straightforward to generalize the gradient operation to posit a **four-gradient** of a scalar function ϕ :

$$\begin{aligned}\partial_\mu\phi &\equiv \frac{\partial\phi}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial\phi}{\partial t}, \nabla\phi\right) , \\ \partial^\mu\phi &\equiv \frac{\partial\phi}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial\phi}{\partial t}, -\nabla\phi\right) .\end{aligned}\tag{72}$$

These are respectively in covariant and contravariant forms, and define true 4-vectors. Observe the shorthand notation of the differentiation. Lorentz transformation of such forms will dictate their use as a compact tool for representing key results in electrodynamics. This is also true in the quantum domain: e.g. the Dirac equation of QED.

- We can also differentiate four-vectors, and provided that we maintain familiar protocols with the indices, the result is either a scalar or a tensor. We usually differentiate a contravariant 4-vector using a covariant derivative, and this is the **four-divergence**:

$$\partial_\mu A^\mu \equiv \frac{\partial A^\mu}{\partial x^\mu} .\tag{73}$$

Such forms appear in a covariant construction of Maxwell's and other electrodynamic equations.

There are four types of integrations that can be constructed using spacetime dimensions. These increase in dimensions. The simplest is just an integral over a curve in four-space, so that the element of integration is just the line element, i.e. dx_α . For the others, our familiarity with 3D spatial integrations, and analogies extend to spacetime. In general, the integration elements are **Jacobians** for the coordinate transformation from the curvilinear surface involved to the projected plane in the restricted dimensions. For example, in 2D, one integrates using a surface element

$$dS_{\alpha\beta} \equiv \begin{vmatrix} dx_\alpha & dx'_\alpha \\ dx_\beta & dx'_\beta \end{vmatrix} = dx_\alpha dx'_\beta - dx'_\alpha dx_\beta ,\tag{74}$$

employing the familiar determinant notation for the Jacobian. N.B. *primes here denote alternative coordinate basis* as opposed to another inertial frame.

In 3D, the surface element projection is also a determinant:

$$dS_{\alpha\beta\gamma} \equiv \begin{vmatrix} dx_\alpha & dx'_\alpha & dx''_\alpha \\ dx_\beta & dx'_\beta & dx''_\beta \\ dx_\gamma & dx'_\gamma & dx''_\gamma \end{vmatrix} . \quad (75)$$

This is a tensor of rank 3, antisymmetric in all indices.

- The last is an integral over a four-dimensional volume,

$$d^4x = dx^0 dx^1 dx^2 dx^3 \equiv c dt dV . \quad (76)$$

This is clearly a scalar due to the compensation between time dilation and length contraction. For this reason, it is often convenient to express 3D integrals as 4D ones with delta functions in the integrands, as needed.

To prove that it is a scalar, i.e. d^4x is a Lorentz invariant, we observe that $d^4x = \mathcal{J} d^4x'$, where the Jacobian for the $x_\mu \rightarrow x'_\mu$ transformation, i.e.

$$\mathcal{J} = \begin{vmatrix} \frac{\partial x_0}{\partial x'_0} & \frac{\partial x_0}{\partial x'_1} & \frac{\partial x_0}{\partial x'_2} & \frac{\partial x_0}{\partial x'_3} \\ \frac{\partial x_1}{\partial x'_0} & \frac{\partial x_1}{\partial x'_1} & \frac{\partial x_1}{\partial x'_2} & \frac{\partial x_1}{\partial x'_3} \\ \frac{\partial x_2}{\partial x'_0} & \frac{\partial x_2}{\partial x'_1} & \frac{\partial x_2}{\partial x'_2} & \frac{\partial x_2}{\partial x'_3} \\ \frac{\partial x_3}{\partial x'_0} & \frac{\partial x_3}{\partial x'_1} & \frac{\partial x_3}{\partial x'_2} & \frac{\partial x_3}{\partial x'_3} \end{vmatrix} . \quad (77)$$

Each of the elements of this determinant is a constant because of the linearity of the form of Lorentz transformations. The determinant can be expressed as a product of simpler determinants corresponding to rotations in two coordinates: *the overall transformation is a sequence of more elementary Lorentz boosts and rotations*. These more elementary rotations in 4D space-time clearly have determinants equal to unity as all rotations do, whether they involve real or imaginary angles. Thus d^4x is a Lorentz invariant under generalized rotations.