1.2 Laplace Transform Applications

The application of Laplace transforms to ordinary differential equations is illustrated by the example of an LRC circuit driven by a square pulse. The circuit is controlled by a switch that is closed at time $t = 0$, and opened at time $t = T$.

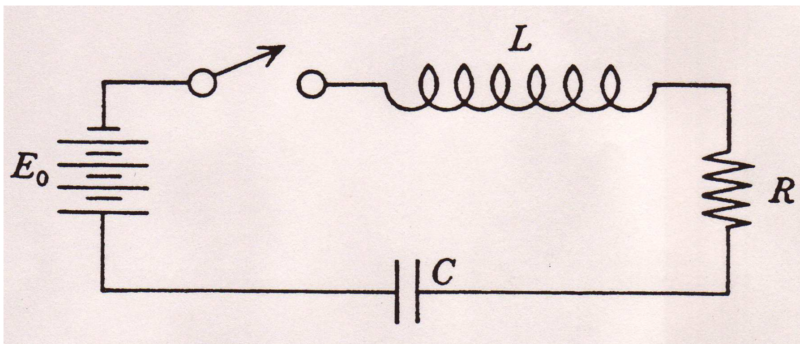

Figure 1: The LRC circuit considered in the illustration of Laplace transform usage, with a potential drop $E_0 \to v(t)$.

The voltage $v(t)$ is described by the integro-differential equation

$$v(t) \; = \; L\frac{di}{dt} + R\,i(t) + \frac{q(t)}{C} \; = \; \begin{cases} 0\ , & t < 0, \\ V_0\ , & 0 < t < T, \\ 0\ , & T < t\ . \end{cases} \tag{23}$$

Here i is the current, q the charge in the circuit, and R is the resistance of the resistor, L is the inductance of the inductor, and C is the capacitance of the capacitor. The charge and the current are related by

$$\frac{dq}{dt} \; = \; i(t) \quad , \tag{24}$$

which when combined with Eq. (23) yields a 2nd order ODE.

Define the Laplace transforms

$$\begin{aligned} I(s) \; &\equiv \; \mathcal{L}\,[i(t)] \quad , \quad Q(s) \; \equiv \; \mathcal{L}\,[q(t)] \quad , \\ V(s) \; &\equiv \; \mathcal{L}\,[v(t)] \; = \; V_0 \int_0^T e^{-st}\,dt \; = \; \frac{V_0}{s}\left(1 - e^{-sT}\right) \quad . \end{aligned} \tag{25}$$

Then the Laplace transforms of Eqs. (23) and (24) yield the simultaneous algebraic equations

$$\begin{aligned} L\left\{s\,I(s) - i(0)\right\} + R\,I(s) + \frac{Q(s)}{C} &= V(s) \\ s\,Q(s) - q(0) &= I(s) \quad . \end{aligned} \tag{26}$$

The initial conditions will be supposed to be $i(0) = 0 = q(0)$, quiescent beginnings. Then

$$\left[Ls + R + \frac{1}{sC}\right] I(s) = \frac{V_0}{s}\left(1 - e^{-sT}\right) \quad , \tag{27}$$

This solves as

$$I(s) = \frac{V_0}{L}\,\frac{1 - e^{-sT}}{(s+\alpha)^2 + \omega^2} \quad , \tag{28}$$

where we have defined

$$\alpha = \frac{R}{2L} \quad , \quad \omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2} \quad . \tag{29}$$

There are three cases for solution, based on the sign of ω^2. While the integration of Eq. (28) could be evaluated using known identities, here the inverse Laplace transforms are used for illustrative purposes.

(a) $\omega^2 > 0$, which corresponds to an **oscillatory** solution. Then $I(s)$ has simple poles at $s = -\alpha \pm i\omega$. To obtain the inverse Laplace transforms, the contour path is chosen as $s = c + iy$, where the constant $c > -\alpha$ places this contour to the right of the poles. Then

$$\begin{aligned} i(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{V_0}{L}\,\frac{1 - e^{-sT}}{(s+\alpha)^2 + \omega^2}\,e^{st}\,ds \\ &= \frac{V_0}{2\pi i\,L}\left[\int_{c-i\infty}^{c+i\infty} \frac{e^{st}\,ds}{(s+\alpha)^2 + \omega^2} - \int_{c-i\infty}^{c+i\infty} \frac{e^{s(t-T)}\,ds}{(s+\alpha)^2 + \omega^2}\right] \quad . \end{aligned} \tag{30}$$

For $t < 0$, both contours must be closed to the right, thus enclosing no poles. Accordingly, $i(t) = 0$ for $t < 0$, as is obviously required.

For $0 < t < T$, the second integral must be closed to the right, as before, and again contributes zero. However, the first integral must be closed to the left, thereby enclosing both poles. Two residues come into play, so that

$$\begin{aligned} i(0<t<T) &= \frac{V_0}{2\pi i\, L}\, 2\pi i \left[\text{Res}(-\alpha+i\omega) + \text{Res}(-\alpha-i\omega) \right] \\ &= \frac{V_0}{L}\left[\frac{e^{(-\alpha+i\omega)t}}{2i\omega} + \frac{e^{(-\alpha-i\omega)t}}{(-2i\omega)} \right] = \frac{V_0}{\omega L}\, e^{-\alpha t}\, \sin\omega t \quad . \end{aligned} \tag{31}$$

This solution displays a ringing oscillation, driven largely by the feedback between the capacitance and the inductance, being damped by the resistance.

For $t > T$, both integrals must be closed to the left, again surrounding both poles. The first integral is identical to that in Eq. (31), while the second is of the same form, but with the substitution $t \to t - T$. Collecting results,

$$i(t) = \frac{V_0}{\omega L}\left[\mathcal{H}(t)\, e^{-\alpha t}\, \sin\omega t - \mathcal{H}(t-T)\, e^{-\alpha(t-T)}\, \sin\omega(t-T) \right] \quad , \tag{32}$$

where

$$\mathcal{H}(t) = \begin{cases} 1\,, & t > 0 \quad , \\ 0\,, & t \le 0 \quad , \end{cases} \tag{33}$$

is the Heaviside step function. The solution therefore exhibits not only a driven ringing, but also a residual one due to the abrupt shutoff of the voltage.

We now illustrate these solutions using `Mathematica` (code below). This oscillatory domain is depicted in Fig. 2, where partial damping is apparent, as is the "negative" response to the switching off the voltage pulse.

```
ioscill[alp_, w_, t_, tt_]:=(
 HeavisideTheta[t] Exp[-alp t] Sin[w t] -
 HeavisideTheta[t - tt] Exp[-alp (t - tt)] Sin[w (t - tt)] )/w

Plot[ ioscill[1.0, 3 Pi/2, t, Pi/2], {t, 0, 2 Pi},
 AxesLabel -> {t, i[t]},
 PlotLabel -> "LRC Current:  oscillatory domain"]
```

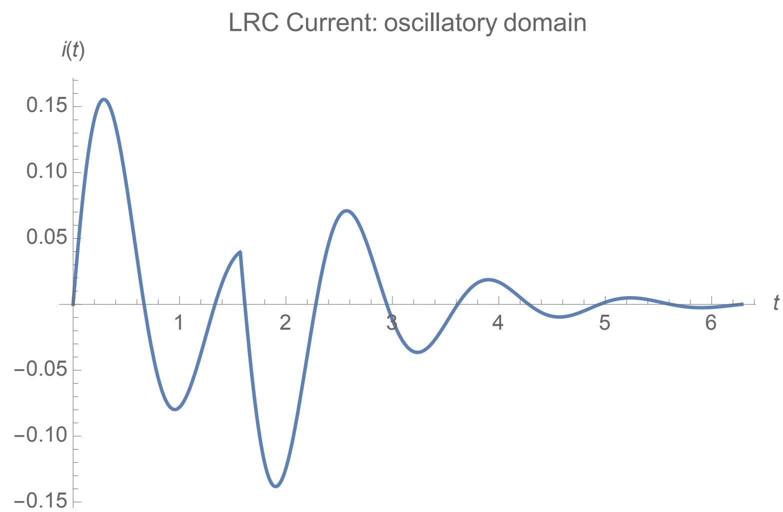

Figure 2: The LRC circuit solution for the current, obtained using Laplace transforms, for the specific case of the oscillatory behavior in Eq. (32).

(b) When $\omega^2 < 0$, corresponding to an **overdamped** case, we can define $\beta^2 = -\omega^2 > 0$ and adapt the previous analysis merely by employing the substitution $\omega \to i\beta$. The result is

$$i(t) \;=\; \frac{V_0}{\beta L}\left[\mathcal{H}(t)\, e^{-\alpha t}\,\sinh\beta t - \mathcal{H}(t-T)\, e^{-\alpha(t-T)}\,\sinh\beta(t-T)\right] \quad . \tag{34}$$

In this case, the resistance dominates the response to the voltage switch and strongly damps out the driver, rendering the capacitance largely ineffective.

This domain of solution behavior is again illustrated using `Mathematica` coding, depicted in Fig. 3. Overwhelming damping is apparent, as is the "negative" response to the switching off the voltage pulse.

```
ioverdamp[alp_, w_, t_, tt_]:=(
  HeavisideTheta[t] Exp[-alp t] Sinh[w t] -
  HeavisideTheta[t - tt] Exp[-alp (t - tt)] Sinh[w (t - tt)] )/w

Plot[ ioscill[1.0, Pi/8, t, Pi/2], {t, 0, 2 Pi},
  AxesLabel -> {t, i[t]},
  PlotLabel -> "LRC Current:  overdamping domain"]
```

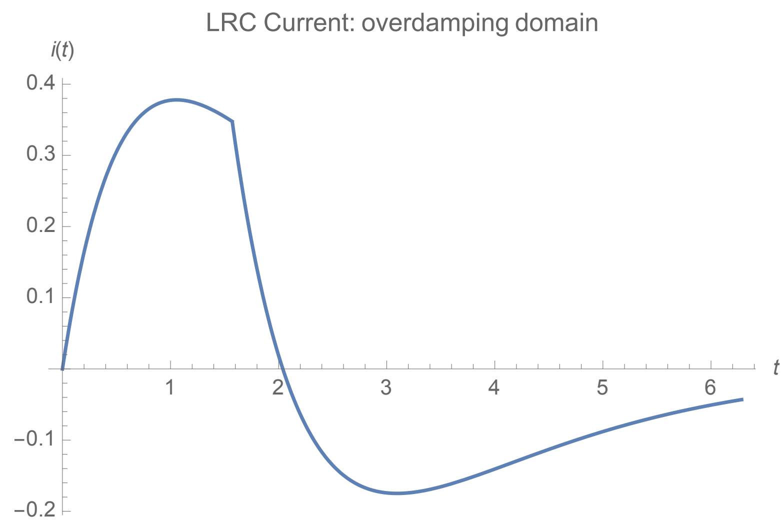

Figure 3: The LRC circuit solution for the current, obtained using Laplace transforms, for the specific case of the overdamped behavior in Eq. (34).

(c) The **critically-damped** case of $\omega^2 = 0$ can be deduced from the above simply by taking the limit $\omega \to 0$ (or $\beta \to 0$), so that

$$i(t) \;=\; \frac{V_0}{L}\left[\Theta(t)\, e^{-\alpha t} - \Theta(t-T)\, e^{-\alpha(t-T)}\right] \quad , \quad \Theta(\tau) \;\equiv\; \tau\mathcal{H}(\tau) \quad . \tag{35}$$

The positive (and shutoff) response is then one of a slowly-driven rise (Spring) followed by a slow damping, never quite achieving oscillatory ringing.

- One can also solve for the charge $q(t)$, which is the integral of the current. In all cases, since there is a simple offset in time because of the gate being opened, the charge asymptotically tends to zero at infinite times, so that no potential is left on the capacitor.

2 Dispersion Relations

The concept of dispersion relations historically arose in physics in the study of optics in material media, with the work of Kramers and Kronig. It is a natural forum for the use of Hilbert transforms since the index of refraction often introduces an inverse dependence on photon frequency, and the light waves can be described by superpositions of complex exponentials. **A&W Sec. 7.2**

2.1 Hilbert Transforms

The Cauchy integral formula for an analytic function $f(z)$ can be specialized to test points along the real axis. Assume that

$$|f(z)| \; < \; a\,|z|^{-\alpha} \quad \text{with} \quad \alpha \; > \; 1 \quad \text{as} \quad |z| \; \to \; \infty \tag{36}$$

to guarantee boundedness in the upper half plane. In this case, the real and imaginary parts of $f(z)$ can be related by integral fomulae that establish Hilbert transform pairs. This development is the integral analog of the Cauchy-Riemann conditions. We start with

$$f(z_0) \; = \; \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\, dz \quad . \tag{37}$$

Now we choose a semi-circular contour closed in the upper half plane, with a straight line segment along the real axis. Then the Residue theorem account for evaluation of the integral in two cases, depending on the sign of the imaginary part of z_0. The real line portions are therefore

$$\begin{aligned} \frac{1}{2\pi i} \int \frac{f(x)}{x - z_0}\, dx \; &= \; f(z_0) \quad , \quad \mathrm{Im}(z_0) \; > \; 0 \\ \frac{1}{2\pi i} \int \frac{f(x)}{x - z_0}\, dx \; &= \; 0 \quad , \quad \mathrm{Im}(z_0) \; < \; 0 \quad . \end{aligned} \tag{38}$$

Next, we let $\mathrm{Im}(z_0) \to 0$. This can be done by considering infinitesimal loops around $z_0 \to x_0$, and the result is (obviously) that the integral approaches the average of the two evaluations in Eq. (38). This is equivalent to considering a semi-circular contour segment C_ϵ encroaching upon x_0.

The **principal value** of the real line integral emerges:

$$f(x_0) \;=\; \frac{1}{\pi i}\,\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\, dx \quad , \tag{39}$$

Then we can take real and imaginary parts, with $f = u + iv$, so that the u and v functions can be related by integral forms, instead of differential ones:

$$\begin{aligned} u(x_0) \;&=\; \frac{1}{\pi}\,\mathcal{P}\int_{-\infty}^{\infty} \frac{v(x)}{x - x_0}\, dx \quad , \\ v(x_0) \;&=\; -\,\frac{1}{\pi}\,\mathcal{P}\int_{-\infty}^{\infty} \frac{u(x)}{x - x_0}\, dx \quad . \end{aligned} \tag{40}$$

Clearly, u and v are **Hilbert transforms** of each other, and knowing either implies knowing the other.

It is common in applications that the function $f(x)$ has an explicit symmetry relation of the form

$$f(-x) \;=\; f^*(x) \quad . \tag{41}$$

For example, if $f(x)$ involves a complex exponential, or a Fourier transform, then this relation applies; i.e. *studies of plane waves and their superpositions* afford this special case. Then, the integrals on the interval $(-\infty,\, 0]$ can be recast using the substitution $x \to -x$ to sample the interval $[0,\, \infty)$, namely via

$$u(-x) \;=\; u(x) \quad , \quad v(-x) \;=\; -v(x) \quad . \tag{42}$$

Then the integral transform pair assumes the form

$$\begin{aligned} u(x_0) \;&=\; \frac{1}{\pi}\,\mathcal{P}\int_{0}^{\infty} v(x) \left(\frac{1}{x - x_0} + \frac{1}{x + x_0} \right) dx \quad , \\ v(x_0) \;&=\; -\,\frac{1}{\pi}\,\mathcal{P}\int_{0}^{\infty} u(x) \left(\frac{1}{x - x_0} - \frac{1}{x + x_0} \right) dx \quad . \end{aligned} \tag{43}$$

i.e.,

$$\begin{aligned} u(x_0) \;&=\; \frac{2}{\pi}\,\mathcal{P}\int_{0}^{\infty} \frac{x\, v(x)}{x^2 - x_0^2}\, dx \quad , \\ v(x_0) \;&=\; -\,\frac{2}{\pi}\,\mathcal{P}\int_{0}^{\infty} \frac{x_0\, u(x)}{x^2 - x_0^2}\, dx \quad . \end{aligned} \tag{44}$$

This was the original form of the **Kramers-Kronig dispersion** relations.

2.2 Optical Dispersion

The classic example of the use of such relations is the transmission of light in material media. Light waves are superpositions of complex exponentials $\exp[i(kx - \omega t)]$. Linearization of the solutions of Maxwell's equations for light propagating in conductive media defines the dielectric properties of the medium:

$$k^2 \;=\; \frac{\omega^2}{c^2}\left(1 + i\frac{4\pi\sigma}{\omega}\right) \quad . \tag{45}$$

Here σ is the medium's electrical conductivity. The appearance of imaginary terms originates in the linear derivatives for the displacement current, etc.

The **refractive index** of the medium $n = ck/\omega$ clearly departs from unity:

$$n \;=\; \left(1 + i\frac{4\pi\sigma}{\omega}\right)^{1/2} \tag{46}$$

Accordingly, light does not travel at c, but at a smaller phase velocity $\omega/k = c/n$. Moreover, the speed depends on ω so that the medium is said to be **dispersive**. In addition, observe that ω is, in general, complex.

Choosing $f \equiv u + iv = n^2 - 1$ as our analytic function, the Kramers-Kronig relations can be written in the form

$$\begin{aligned} \mathrm{Re}\left[n^2(\omega_0) - 1\right] \;&=\; \frac{2}{\pi}\,\mathcal{P}\int_0^\infty \frac{\omega\,\mathrm{Im}\left[n^2(\omega) - 1\right]}{\omega^2 - \omega_0^2}\,d\omega \quad , \\ \mathrm{Im}\left[n^2(\omega_0) - 1\right]) \;&=\; -\frac{2}{\pi}\,\mathcal{P}\int_0^\infty \frac{\omega_0\,\mathrm{Re}\left[n^2(\omega) - 1\right]}{\omega^2 - \omega_0^2}\,d\omega \quad . \end{aligned} \tag{47}$$

The real part describes the deviation of the phase speed from c. The imaginary part then marks imaginary contributions to ω for real k. In the complex exponential $e^{i\omega t}$, these then *define growth or decay (damping) influences of the plasma on the wave.*

- Growth $\Rightarrow$ instability, while decay $\Rightarrow$ dissipative absorption.

• **Example 1:** The principal values encountered in dispersion relation integrals offer potential difficulties for numerical evaluation. This example illustrates how to develop an algorithm to evaluate

$$I \;=\; \mathcal{P}\int_0^{\infty} \frac{f(\omega)\, d\omega}{\omega^2 - \omega_0^2} \tag{48}$$

numerically. The trick is to transform the integral in the neighborhood of $\omega = \omega_0$:

$$I \;=\; \Delta + \int_0^{\omega_0(1-\epsilon)} \frac{f(\omega)\, d\omega}{\omega^2 - \omega_0^2} + \int_{\omega_0(1+\epsilon)}^{\infty} \frac{f(\omega)\, d\omega}{\omega^2 - \omega_0^2} \tag{49}$$

for

$$\Delta \;=\; \mathcal{P}\int_{\omega_0(1-\epsilon)}^{\omega_0(1+\epsilon)} \frac{f(\omega)\, d\omega}{\omega^2 - \omega_0^2} \quad . \tag{50}$$

The principal value can formally be written as a limiting value:

$$\Delta \;=\; \lim_{\delta\to 0}\left\{ \int_{\omega_0(1-\epsilon)}^{\omega_0(1-\delta)} \frac{f(\omega)\, d\omega}{\omega^2 - \omega_0^2} + \int_{\omega_0(1+\delta)}^{\omega_0(1+\epsilon)} \frac{f(\omega)\, d\omega}{\omega^2 - \omega_0^2} \right\} \quad . \tag{51}$$

Now we rescale the integrands by setting $\omega = \omega_0(1-x)$ and $\omega = \omega_0(1+x)$ in the respective integrals. Then

$$\begin{aligned} \Delta \;&=\; \lim_{\delta\to 0}\left\{ \frac{1}{\omega_0}\int_{\delta}^{\epsilon} \frac{f(\omega_0[1+x])\, dx}{x(2+x)} - \frac{1}{\omega_0}\int_{\delta}^{\epsilon} \frac{f(\omega_0[1-x])\, dx}{x(2-x)} \right\} \\ &=\; \frac{1}{\omega_0}\int_0^{\epsilon} \frac{dx}{x}\left\{ \frac{f(\omega_0[1+x])}{2+x} - \frac{f(\omega_0[1-x])}{2-x} \right\} \quad . \end{aligned} \tag{52}$$

For functions $f(\omega)$ that are smooth at $\omega = \omega_0$, the factor in curly braces scales as x as $x \to 0$, and the Δ integration is amenable to numerical techniques. A Taylor series expansion for f about ω_0 can be employed to ascertain the analytic limit of Δ/ϵ as $\epsilon \to 0^+$:

$$\lim_{\epsilon\to 0^+} \frac{\Delta}{\epsilon} \;=\; f'(\omega_0) - \frac{f(\omega_0)}{2\omega_0} \quad\Rightarrow\quad \Delta \;\to\; 0 \quad \text{as} \quad \epsilon \;\to\; 0^+ \quad . \tag{53}$$

This serves as a useful check: in practice, one can never take $\epsilon \to 0$ since this just reintroduces the numerical divergence issues.