To prove Parseval's Theorem, we make use of the integral identity for the Dirac delta function.

$$\begin{aligned}
\int_{-\infty}^{\infty} \left| f(x) \right|^2 dx &= \int_{-\infty}^{\infty} f(x)\, f^*(x)\, dx \\
&= \int_{-\infty}^{\infty} dx \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)\, e^{ixs}\, ds \right\} \\
&\qquad \times \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g^*(s')\, e^{-ixs'}\, ds' \right\} \qquad (35) \\
&= \int_{-\infty}^{\infty} ds\, g(s) \int_{-\infty}^{\infty} ds'\, g^*(s')\, \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix(s-s')}\, dx \\
&= \int_{-\infty}^{\infty} ds\, g(s) \int_{-\infty}^{\infty} ds'\, g^*(s')\, \delta(s-s') \;=\; \int_{-\infty}^{\infty} \left| g(s) \right|^2 ds \quad .
\end{aligned}$$

• **Example 4:** The Fourier transform of a Gaussian is another Gaussian. To demonstrate this, we start with

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$$f(x) \;=\; \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/2\sigma^2} \quad , \qquad (36)$$

of unit normalization, and a width (standard deviation) $\Delta x \sim \sigma$. Then

$$\begin{aligned}
g(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ixs}\, dx \\
&= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2 - ixs}\, dx \\
&= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2\sigma^2} \left[(x+\sigma^2 is)^2 + \sigma^4 s^2 \right] \right\} dx \qquad (37) \\
&= \frac{1}{2\pi\sigma}\, e^{-\sigma^2 s^2/2} \int_{-\infty+i\sigma^2 s}^{\infty+i\sigma^2 s} e^{-z^2/2\sigma^2}\, dz \quad , \quad z \;=\; x + i\sigma^2 s \\
&= \frac{1}{\sqrt{2\pi}}\, e^{-\sigma^2 s^2/2} \quad ,
\end{aligned}$$

using the Residue theorem to evaluate the integral of the Gaussian by equating it to one along the real axis (there are no poles for the Gaussian). Observe that this Fourier transform has a width $\Delta s \sim 1/\sigma$ so that we deduce the correlation $\Delta x \Delta s \sim 1$ relating the widths of the Gaussian and its transform.

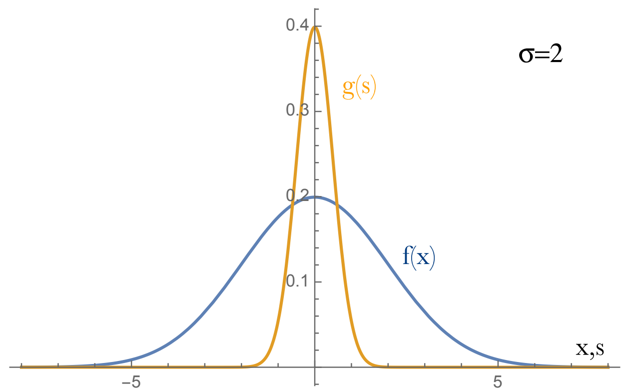

Figure 6: A Gaussian $f(x)$ of standard devision $\sigma = 2$ (blue) and its Fourier transform $g(s)$ of width $1/\sigma$ (orange).

• This focuses upon an essential mathematical property of Fourier transforms: the product of the **variances** (i.e. σ^2) of a function $f(x)$ and its Fourier transform is of the order of unity.

∗ Broad functions $f(x)$ have narrow Fourier transforms and vice versa. The property essentially follows from constructive or destructive contributions to the integrals involving complex exponentials.

• This property underpins various important physics elements that connect to superpositions of infinite or finite plane waves and therefore are intimately related to Fourier transforms: **A&W pp. 940-1**

∗ **Heisenberg's uncertainty principle** relating conjugate variables in quantum mechanics (e.g. $\Delta x \Delta p \gtrsim \hbar$ for de Broglie wavelengths $\lambda = \hbar/p$);

∗ classical radiation power spectra in Larmor formalism for accelerating charges in electrodynamics (e.g. $\Delta x \Delta \omega \gtrsim c$). Examples include Thomson scattering, bremsstrahlung and synchrotron radiation;

∗ a myriad of elements relating to hydromagnetic waves in turbulent plasmas, such as magnetic field turbulence spectra and diffusion of charges using classical electrodynamics in turbulent environs.

∗ acoustic mode contexts such as quasi-periodic oscillations in neutron star crusts, i.e. outer layers.

• **Example 4:** To further illustrate this "uncertainty principle" property of Fourier transforms, we consider

$$f(x) \;=\; \frac{1}{\pi}\,\frac{a}{a^2+x^2} \quad , \tag{38}$$

a Lorentzian with width a and peak amplitude $1/a$. Its transform is

$$g(s) \;=\; \frac{a}{\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-ixs}\,dx}{(x+ia)\,(x-ia)} \quad , \tag{39}$$

which is easily evaluated using the Cauchy residue theorem.

∗ For $s > 0$, close the contour with the semicircle in the *lower* half plane so that it captures the pole at $z = -ia$, giving $g(s>0) = e^{-as}/\sqrt{2\pi}$.

∗ For $s < 0$, close the contour instead with the semicircle in the *upper* half plane, capturing the pole at $z = ia$; this gives $g(s<0) = e^{+as}/\sqrt{2\pi}$.

• For $s = 0$, the integral evaluates simply as π/a using the arctan function. Combining results, we have

$$g(s) \;=\; \frac{e^{-a|s|}}{\sqrt{2\pi}} \quad , \tag{40}$$

which has a width $1/a$. Hence the $\Delta x \Delta s \sim 1$ character is established.

∗ Logically inverting this problem, introducing a decay lifetime to a plane wave state via an exponential $e^{-a|s|}$ imposes a Lorentz profile to the spectra of discrete radiative transitions in physics (**natural line broadening**).

∗ Observe that the discontinuous derivative at $s = 0$ is related to the fact that $f(x)$ does not approach zero sufficiently fast as $x \to \pm\infty$.

7. INTEGRAL TRANSFORMS AND DISPERSION RELATIONS

Matthew Baring — Lecture Notes for PHYS 516, Fall 2022

1 Various Integral Transforms

The concept of the Fourier transform can be extended to treat more general weightings in the integrands that are useful for different contexts. For a function $f(x)$, if

$$g(s) \; = \; \int_a^b f(x)\, K(s,\, x)\, dx \qquad (1)$$

exists, it is called the **integral transform** of $f(x)$ by the **kernel** $K(s,\, x)$. Common examples are as follows:

$$\begin{aligned}
\text{Fourier transform} \quad g(s) \; &= \; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ixs}\, dx \\
\text{Laplace transform} \quad g(s) \; &= \; \int_0^{\infty} f(x)\, e^{-xs}\, dx \\
\text{Hankel transform} \quad g(s) \; &= \; \int_0^{\infty} f(x)\, x\, J_n(xs)\, dx \qquad (2) \\
\text{Mellin transform} \quad g(s) \; &= \; \int_0^{\infty} f(x)\, x^{s-1}\, dx \\
\text{Hilbert transform} \quad g(s) \; &= \; \frac{1}{\pi}\, \mathcal{P} \int_{-\infty}^{\infty} \frac{f(x)}{x-s}\, dx
\end{aligned}$$

Note that all are linear transforms. Note also that $\mathcal{P}$ denotes the *Cauchy principal value* of the integral that treats cancelling divergences around the singularity in the integrand; this subtlety will be explored more in Section 2.

• Now let us explore the **Laplace transform**, and its relation to the Fourier transform. In cases where $f(x)$ is not integrable over $(-\infty, \infty)$, we can truncate the integration range by applying a *convergence factor* $H(x)\, e^{-cx}$ where $c > 0$ is real and $H(x)$ is the **Heaviside step function**:

A&W Sec. 15.8

$$H(x) \;=\; \begin{cases} 0 & , \quad x \;<\; 0\,, \\ 1 & , \quad x \;>\; 0\,. \end{cases} \tag{3}$$

Then, provided that $f(x)$ is of exponential order $c_1 \;<\; c$, meaning that $f(x)$ grows no faster than $e^{c_1 x}$ as $x \to \infty$, then the Fourier transform of $\sqrt{2\pi} f(x) H(x) e^{-cx}$ is called the Laplace transform of $f(x)$:

$$g(y) \;=\; \int_{-\infty}^{\infty} f(x)\, H(x)\, e^{-cx}\, e^{-ixy}\, dx \;=\; \int_0^{\infty} f(x)\, e^{-(c+iy)x}\, dx \quad . \tag{4}$$

It is conventional to define $s \;=\; c + iy$ so that the Laplace transform is effectively a Fourier transform rotated and translated in the complex plane. Then

$$g(s) \;=\; \int_0^{\infty} f(x) e^{-sx}\, dx \quad . \tag{5}$$

The inverse of this can be determined using the inverse Fourier transform:

$$f(x)\, H(x)\, e^{-cx} \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} g(y)\, e^{ixy}\, dy \;=\; \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s)\, e^{(s-c)x}\, ds \tag{6}$$

so that the **Laplace inversion integral** becomes

$$f(x)\, H(x) \;=\; \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s)\, e^{sx}\, ds \quad . \tag{7}$$

• Note that the constant c in this contour integral must be chosen so that the contour is to the right of any poles of $g(s)$. Then, for $x \;<\; 0$ the contour can be closed with a semicircle to the right half plane, correctly giving $H(x) f(x) = 0$. In contrast, for $x > 0$, we complete the contour in the left half plane, capturing all the poles of $g(s)$ and giving

$$f(x > 0) \;=\; \sum_p \text{Res}\Big[g(s)\Big]_{s=s_p} \quad . \tag{8}$$

• Simple examples of Laplace transforms include

$$f(x) \;=\; x^n \quad\Rightarrow\quad \mathcal{L}\,[f(x)] \;=\; \frac{n!}{s^{n+1}} \;\;, \quad \mathrm{Re}(s) \;>\; 0 \;\; . \tag{9}$$

for integer n (observe the new Laplace transform notation), and

$$f(x) \;=\; e^{ax} \quad\Rightarrow\quad \mathcal{L}\,[f(x)] \;=\; \int_0^\infty e^{(a-s)x}\,dx \;=\; \frac{1}{s-a} \;\;, \quad \mathrm{Re}(s) \;>\; a \;\; . \tag{10}$$

Another example that is germane to electrical circuit theory (see Section 2) or other systems with sinusoidal response is

$$\begin{aligned} f(x) \;&=\; \sin ax \quad\Rightarrow\quad \mathcal{L}\,[f(x)] \;=\; \frac{a}{a^2+s^2} \;\;, \quad \mathrm{Re}(s) \;>\; 0 \;\; , \\ f(x) \;&=\; \cos ax \quad\Rightarrow\quad \mathcal{L}\,[f(x)] \;=\; \frac{s}{a^2+s^2} \;\;, \quad \mathrm{Re}(s) \;>\; 0 \;\; . \end{aligned} \tag{11}$$

These results are most simply demonstrated by expressing the trigonometric functions in terms of complex exponentials.

Each of the highlighted transforms has particular situations where it is useful:

- **Bessel** or Hankel transforms can be applied to cylindrical systems of partial differential equations since often such geometry sets up Bessel function character in the dimension orthogonal to the cylinder axis.
- **Mellin** transforms enhance the facility of treating problems with power-law (scale-independent) character.
- **Hilbert** transforms are germane to dispersion theory, and will be explored further in Section 2 below.

• **Example 1:** Consider the Mellin transform of a truncated power-law function $P(k) = P_0\,k^{-\gamma}$ for $k_{\rm min} < k < k_{\rm max}$ and $P(k) = 0$ outside this range. Such distributions are commonly encountered in turbulence theory (including sandpiles, seismic activity), and are scale-independent (power-law) between the stirring scale $k_{\rm min}$ and the dissipation scale $k_{\rm max}$. In such cases, $k = 2\pi/\lambda$ represents the wavenumber of the turbulence, with λ being its wavelength. The Mellin transform for this distribution is also power-law:

$$g(s) \;=\; \int_0^\infty P(k)\,k^{s-1}\,dk \;=\; \frac{k_{\rm max}^{s-\gamma} - k_{\rm min}^{s-\gamma}}{s-\gamma} \quad . \tag{12}$$

1.1 Generic Properties

Here we summarize a handful of integral transform properties that prove useful in manipulating differential and even integral equations. To facilitate compact forms, we use the following notation:

$$\mathcal{F}\left[f(x)\right] \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ixy}\, dx \tag{13}$$

as the exponential Fourier transform, and

$$\mathcal{L}\left[f(x)\right] \;=\; \int_{0}^{\infty} f(x)\, e^{-sx}\, dx \tag{14}$$

as the Laplace transform of $f(x)$. These transforms display **linearity**:

$$\begin{aligned} \mathcal{F}\left[a_1 f_1(x) + a_2 f_2(x)\right] &= a_1\, \mathcal{F}\left[f_1(x)\right] + a_2 \mathcal{F}\left[f_2(x)\right] \quad , \\ \mathcal{L}\left[a_1 f_1(x) + a_2 f_2(x)\right] &= a_1\, \mathcal{L}\left[f_1(x)\right] + a_2 \mathcal{L}\left[f_2(x)\right] \quad . \end{aligned} \tag{15}$$

The expressions for **derivatives** can be established using integration by parts:

$$\begin{aligned} \mathcal{F}\left[f'(x)\right] &= iy\, \mathcal{F}\left[f(x)\right] \quad , \\ \mathcal{L}\left[f'(x)\right] &= s\, \mathcal{L}\left[f(x)\right] - f(0) \quad . \end{aligned} \tag{16}$$

These forms convert differential equations into algebraic ones, an example of which will be explored in the next Subsection. **Integrals** can be manipulated in similar fashion:

$$\begin{aligned} \mathcal{F}\left[\int_0^x f(x')\, dx'\right] &= \frac{1}{iy}\, \mathcal{F}\left[f(x)\right] \quad , \\ \mathcal{L}\left[\int_0^x f(x')\, dx'\right] &= \frac{1}{s}\, \mathcal{L}\left[f(x)\right] \quad , \end{aligned} \tag{17}$$

which facilitate conversion of integral equations into algebraic ones. These results are established by integration by parts, with the residual terms contributing zero in each case.

Translation formulae are routinely derived:

$$\begin{aligned} \mathcal{F}\left[f(x+a)\right] &= e^{iay}\,\mathcal{F}\left[f(x)\right] \quad , \\ \mathcal{L}\left[f(x+a)\right] &= e^{as}\left\{\mathcal{L}\left[f(x)\right] - \mathcal{H}(a)\int_0^a e^{-sx}\,f(x)\,dx\right\} \quad , \end{aligned} \tag{18}$$

where the second terms vanishes for $a < 0$ because $\mathcal{H}(a)$ is a step function that is unity for positive a, and zero otherwise. **Multiplication by powers of x** is tantamount to differentiation of the transforms:

$$\begin{aligned} \mathcal{F}\Big[x\,f(x)\Big] &= i\,\frac{d}{dy}\,\mathcal{F}\Big[f(x)\Big] \quad , \\ \mathcal{L}\Big[x\,f(x)\Big] &= -\frac{d}{ds}\,\mathcal{L}\Big[f(x)\Big] \quad . \end{aligned} \tag{19}$$

The final class of integral transform properties that will be listed here pertains to **convolutions** of functions, which are defined via

A&W Sec. 15.11

$$f_1 * f_2 \equiv \int_{-\infty}^{\infty} f_1(y)\,f_2(x-y)\,dy \quad , \tag{20}$$

and represent weighted averages of either contributing function. Then, the following **convolution theorems** apply:

$$\begin{aligned} \mathcal{F}\Big[f_1 * f_2\Big] &= \sqrt{2\pi}\,\mathcal{F}\Big[f_1(x)\Big] \times \mathcal{F}\Big[f_2(x)\Big] \quad , \\ \mathcal{L}\Big[f_1 * f_2\Big] &= \mathcal{L}\Big[f_1(x)\Big] \times \mathcal{L}\Big[f_2(x)\Big] \quad . \end{aligned} \tag{21}$$

These can be deployed to expedite evaluation of individual transforms if the transform of the convolution is simply determined. Coupled with these, there are the **converse convolution theorems**. If, for $i = 1, 2$, $\mathcal{F}_i(y) = \mathcal{F}\left[f_i(x)\right]$ and $\mathcal{L}_i(y) = \mathcal{L}\left[f_i(x)\right]$ define the transforms, then

$$\begin{aligned} \mathcal{F}\Big[f_1 \times f_2\Big] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathcal{F}_1(z)\,\mathcal{F}_2(y-z)\,dz \quad , \\ \mathcal{L}\Big[f_1 \times f_2\Big] &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{L}_1(z)\,\mathcal{L}_2(s-z)\,dz \quad . \end{aligned} \tag{22}$$

In other words, the Fourier (Laplace) transform of a product is proportional to the convolution of the Fourier (Laplace) transforms, and *vice versa*.

Fourier Convolution Theorems

Here are the proofs for the Fourier convolution theorems:

$$
\begin{aligned}
\mathcal{F}\Big[f_1 * f_2\Big] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx}\, dx \int_{-\infty}^{\infty} f_1(y)\, f_2(x-y)\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(y)\, e^{-isy}\, dy \int_{-\infty}^{\infty} f_2(x-y)\, e^{-is(x-y)}\, dx \\
&= \sqrt{2\pi}\, \mathcal{F}\Big[f_1(x)\Big] \times \mathcal{F}\Big[f_2(x)\Big] \quad ,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{F}_1(z)\, \mathcal{F}_2(y-z)\, dz &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} f_1(t)\, e^{-izt}\, dt \int_{-\infty}^{\infty} f_2(x)\, e^{-i(y-z)x}\, dx \\
&= \int_{-\infty}^{\infty} f_1(t)\, dt \int_{-\infty}^{\infty} f_2(x)\, e^{-ixy}\, dx\, \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x-t)z}\, dz \\
&= \int_{-\infty}^{\infty} f_1(t)\, dt \int_{-\infty}^{\infty} f_2(x)\, e^{-ixy}\; \delta(x-t)\, dx \\
&= \int_{-\infty}^{\infty} f_1(t)\, f_2(t)\, e^{-ity} \;\equiv\; \sqrt{2\pi}\, \mathcal{F}\Big[f_1 \times f_2\Big] \quad .
\end{aligned}
$$