• **Example 1:** The *square wave* or step function is formally discontinuous, but has a well-defined Fourier series. Consider the form

A&W pp. 892–3

$$f(\theta) \;=\; \begin{cases} +1 \;, & 0 \;<\; \theta \;<\; \pi \quad , \\ 0 \;, & \theta = \pi, \\ -1 \;, & \pi \;<\; \theta \;<\; 2\pi \quad . \end{cases} \tag{9}$$

Then

$$a_n \;=\; \frac{1}{\pi}\int_0^{\pi} \cos n\theta \, d\theta - \frac{1}{\pi}\int_{\pi}^{2\pi} \cos n\theta \, d\theta \;=\; \left.\frac{\sin n\theta}{n\pi}\right|_0^{\pi} - \left.\frac{\sin n\theta}{n\pi}\right|_{\pi}^{2\pi} \;=\; 0 \quad . \tag{10}$$

Likewise

$$b_n \;=\; \frac{1}{\pi}\int_0^{\pi} \sin n\theta \, d\theta - \frac{1}{\pi}\int_{\pi}^{2\pi} \sin n\theta \, d\theta \;=\; -\left.\frac{\cos n\theta}{n\pi}\right|_0^{\pi} + \left.\frac{\cos n\theta}{n\pi}\right|_{\pi}^{2\pi} \quad , \tag{11}$$

so that

$$b_n \;=\; \frac{2}{n\pi}\Big[1 - (-1)^n\Big] \;=\; \begin{cases} \dfrac{4}{n\pi} \;, & n = 1, 3, 5, 7 \,\ldots \\ 0 \;, & n = 0, 2, 4, 6, \,\ldots \end{cases} \tag{12}$$

It then follows that the Fourier series for the square wave $f(\theta)$ is

$$f(\theta) \;=\; \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)\theta}{2k+1} \quad , \quad 0 \;<\; \theta \;<\; 2\pi \quad . \tag{13}$$

• This particular example exhibits a pathological feature known as **Gibb's phenomenon**: the Fourier series does not converge uniformly in the neighborhood of a discontinuity.

A&W Sec. 14.5

∗ The amplitude of the "overshoot" remains finite even as the number of terms N computed tends to infinity, although the width of the overshoot tends to zero (as $1/N$) as $N \to \infty$.

• **Example 2 (Numerical):** The *square wave* provides an ideal case to exhibit Gibb's phenomenon. This is simply done using `Mathematica` coding. Eq. (13) can be truncated to a series of N terms via

```
S[x_, N_] := (4/Pi) Sum[ Sin[(2k+1) x]/(2k+1), {k, 0, N}]
```

Results for different N on $0 < \theta, \pi$, in increasing order, are now illustrated, to clearly highlight Gibb's phenomenon; the $[\pi, 2\pi]$ interval is a simple upside-down inversion of these.

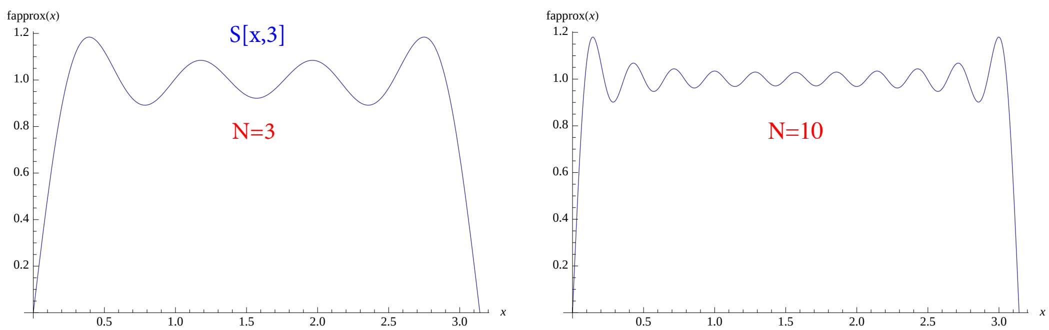

Figure 1: The truncated sums $S[x, 3]$ (left) and $S[x, 10]$ (right) used to describe Eq. (13), plotted in `Mathematica` using the command `Plot[S[x, N], {x, 0, Pi}, AxesLabel -> {x, fapprox[x]}, PlotRange -> Full]`.

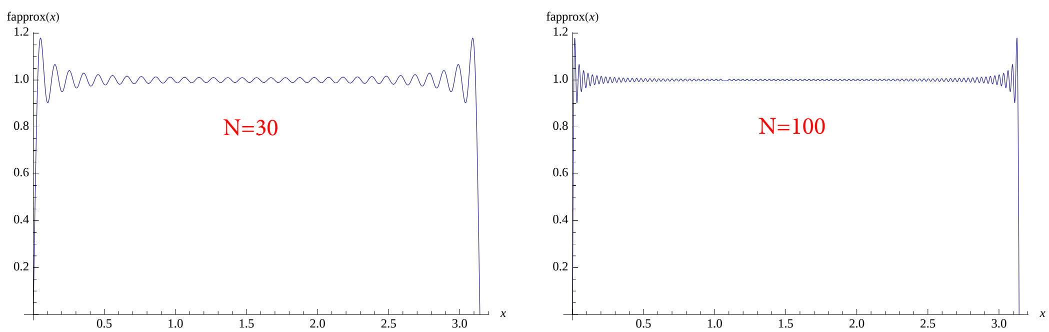

Figure 2: The truncated sums $S[x, 30]$ (left) and $S[x, 100]$ (right) that clearly depict the non-uniform convergence of Gibb's phenomenon for the square wave in the vicinity of its discontinuities.

• The error incurred by approximating the discontinuity with a sum of continuous functions is maximized at around the 17% level.

• **Example 3:** Consider the Fourier series for the parabola $f(x) = x^2$ on the interval $|x| \leq \pi$. The function is even, and the Fourier coefficients are routinely determined: **A&W Ex. 14.3.3**

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx \;=\; \frac{2\pi^2}{3} \quad , \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,\cos nx\,dx \;=\; (-1)^n\,\frac{4}{n^2} \quad . \end{aligned} \tag{14}$$

From this we derived the Fourier series

$$x^2 \;=\; \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\,\frac{\cos nx}{n^2} \quad . \tag{15}$$

Setting $x = \pi$ sets $\cos n\pi = (-1)^n$ so that we can obtain the value of the Riemann zeta function:

$$\zeta(2) \;\equiv\; \sum_{n=1}^{\infty}\frac{1}{n^2} \;=\; \frac{\pi^2}{6} \quad . \tag{16}$$

For $x = 0$, we also get

$$\eta(2) \;\equiv\; \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} \;=\; \frac{\pi^2}{12} \quad . \tag{17}$$

• For a periodic function $f(x)$ with period $L \neq 2\pi$, we replace θ in the foregoing analysis by $2\pi x/L$. Then

$$f(\theta) \;=\; \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n\cos\frac{2n\pi x}{L} + B_n\sin\frac{2n\pi x}{L}\right) \quad . \tag{18}$$

with

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\,\cos\frac{2n\pi x}{L}\,dx \\ B_n &= \frac{2}{L}\int_0^L f(x)\,\sin\frac{2n\pi x}{L}\,dx \quad . \end{aligned} \tag{19}$$

This connects to more real-life systems such as modeling crystal lattices.

1.1 Complex Form for Fourier Series

It is often much more convenient and necessary to carry along the information of both sinusoidal components in Fourier series. This is expedited using complex exponential forms. Using Euler's formula $e^{\pm in\theta} = \cos n\theta \pm i \sin n\theta$, our Fourier series in Eq. (1) can be rewritten

$$f(\theta) \; = \; \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \, \frac{e^{in\theta} + e^{-in\theta}}{2} + b_n \, \frac{e^{in\theta} - e^{-in\theta}}{2i} \right) \quad . \tag{20}$$

Collecting terms, and relabelling those with negative exponents gives the complex exponential form

$$f(\theta) \; = \; \sum_{n=-\infty}^{\infty} c_n \, e^{in\theta} \quad , \quad c_n \; = \; \frac{1}{2} \begin{cases} a_n - ib_n \quad , & n \; > \; 0 \quad , \\ a_0 \quad , & n \; = \; 0 \quad , \\ a_n + ib_n \quad , & n \; < \; 0 \quad . \end{cases} \tag{21}$$

The **orthogonality property** of complex exponentials is

$$\int_0^{2\pi} e^{in\theta} e^{-im\theta} \, d\theta \; = \; 2\pi \delta_{nm} \quad , \tag{22}$$

and can be derived by direct integration, or from Eqs. (3) and (4). By analogy with the trigonometric function analysis, we form complex exponential integral moments to efficiently deduce compact forms for the Fourier coefficients:

$$c_n \; = \; \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \, e^{-in\theta} \, d\theta \quad . \tag{23}$$

This result could be also determined by substitution of complex exponentials into the trigonometric forms for a_n and b_n, but that path is less efficient.

- Observe the "orthonormality" property of

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 \, d\theta \; &= \; \frac{1}{2\pi} \int_0^{2\pi} d\theta \left(\sum_{n=-\infty}^{\infty} c_n e^{in\theta} \right) \left(\sum_{m=-\infty}^{\infty} c_m^* e^{-im\theta} \right) \\ &= \; \sum_{n,m} c_n c_m^* \, \delta_{nm} \; = \; \sum_{n=-\infty}^{\infty} |c_n|^2 \quad . \end{aligned} \tag{24}$$

This is a discrete (i.e. series) analog of **Parseval's theorem**, and provides useful information for power spectra of turbulent systems.

2 The Fourier Transform

The **Fourier transform** is the continuum limit of the complex Fourier series as the (spatial) period $L \to \infty$. This corresponds to small changes in the series terms between neighboring values of n. Consider **A&W Sec. 15.2**

$$f(x) \;=\; \sum_{n=-\infty}^{\infty} c_n \, e^{i\,2\pi n x/L} \quad , \quad c_n \;=\; \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-i\,2\pi n x/L}\, dx \quad . \tag{25}$$

To map over to the continuum limit, we define $s = 2\pi n/L$ and the Fourier transform $Lc_n = \sqrt{2\pi} g(s)$, using the correspondence

$$\sum_{n=-\infty}^{\infty} \mathcal{F}(n) \;\to\; \int_{-\infty}^{\infty} \mathcal{F}(n)\, dn \;\to\; \frac{L}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(s)\, ds \tag{26}$$

for the Fourier series, with $\mathcal{F}(n)$ representing each term of the series. Then

$$f(x) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)\, e^{ixs}\, ds \tag{27}$$

defines the **inverse Fourier transform** of the function $g(s)$, and

$$g(s) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ixs}\, dx \tag{28}$$

defines the **Fourier transform** of the function $f(x)$. These follow simply from inserting the correspondence Eq. (26) into Eq. (25).

$*$ Observe that normalization conventions with the integrals defining Fourier transform/inverse pairs are not unique. For example, often, the factor outside one of the integrals in the pair is $1/2\pi$, while the other is unity.

$*$ Observe that a sufficient (but not necessary) condition for convergence of the Fourier transform is that $f(x)$ is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} \Big| f(x) \Big|\, dx \quad \text{exists.} \tag{29}$$

• Combining Eqs. (27) and (28) yields

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} ds\, e^{ixs} \int_{-\infty}^{\infty} dx'\, f(x')\, e^{-ix's} \\
&= \int_{-\infty}^{\infty} dx'\, f(x')\, \frac{1}{2\pi}\int_{-\infty}^{\infty} ds\, e^{i(x-x')s}
\end{aligned}
\tag{30}
$$

after re-ordering the integrals. Then we introduce the **Dirac delta function** $\delta(x)$ via the definition **A&W pp. 937–8**

$$
\delta(x) = 0 \quad , \quad x \neq 0 \quad \text{for} \quad \int_a^b \delta(x)\, dx = 1 \quad , \quad a < 0 < b \quad . \tag{31}
$$

It follows that Eq. (30) establishes the integral identity for the Dirac delta function:

$$
\delta(\chi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds\, e^{i\chi s} \quad , \tag{32}
$$

i.e., the destructive interference of the integrand reduces the integral to zero when $\chi \neq 0$, and it is simply divergent for $\chi = 0$.

• Note that $\delta(x)$ is a distribution, or a **generalized function**. It can be expressed as the limit form of a sequence $\delta_n(x)$ of normal functions.

$$
\begin{aligned}
\delta_n(x) &= \frac{n}{\pi}\,\frac{1}{1+n^2x^2} \\
\text{or}\quad \delta_n(x) &= \frac{n}{\sqrt{\pi}}\, e^{-n^2x^2} \\
\text{or}\quad \delta_n(x) &= \frac{1}{n\pi}\,\frac{\sin^2 nx}{x^2}
\end{aligned}
\tag{33}
$$

provide some alternatives, all of which approach $\delta(x)$ as $n \to \infty$. They have a peak magnitude $\propto n$ and peak width $\propto 1/n$, and all have unit area on the interval $(-\infty, \infty)$. This behavior can be exhibited using the `Mathematica` coding for the functions:

```
delta1[n_, x_] := n/Pi/(1+ n^2 x^2 )
delta2[n_, x_] := n/Sqrt[Pi] * Exp[ -n^2 x^2]
delta3[n_, x_] := (Sin[ n x])^2/(n Pi x^2)
```

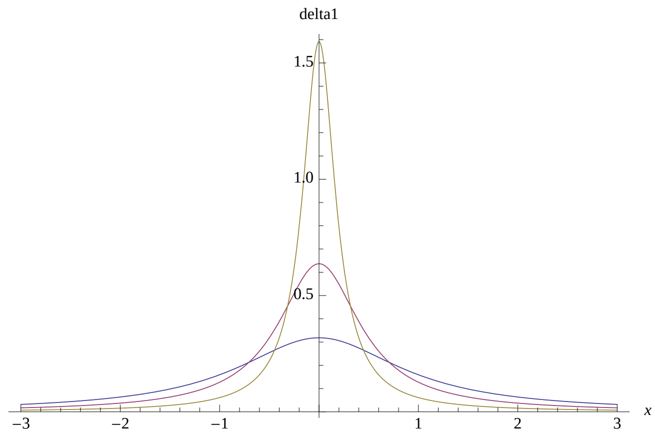

Figure 3: The sequence of functions $\delta_n(x) = n/\pi/(1+n^2x^2)$, for $n = 1, 2, 5$ used to approximate the Dirac δ function when high values of n are adopted.

`Mathematica` plots of these are acquired using the commands

```
Plot[ {delta1[1,x], delta1[2,x], delta1[5,x]}, {x, -3, 3},
      AxesLabel -> {x, delta1}, PlotRange -> Full]
Plot[ {delta2[1,x], delta2[2,x], delta2[5,x]}, {x, -3, 3},
      AxesLabel -> {x, delta2}, PlotRange -> Full]
Plot[ {delta3[1,x], delta3[2,x], delta3[5,x]}, {x, -3, 3},
      AxesLabel -> {x, delta3}, PlotRange -> Full]
```

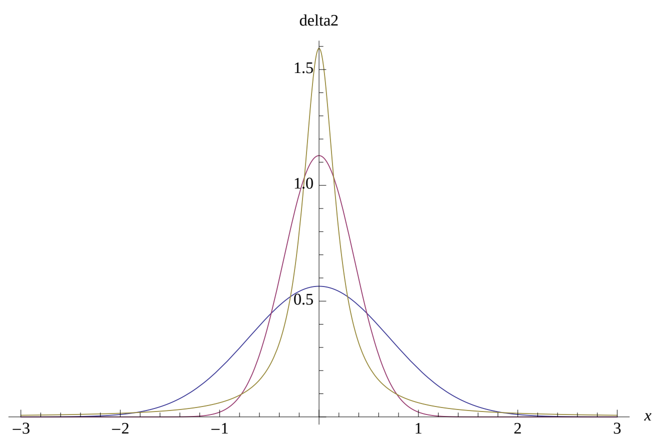

Figure 4: The sequence of functions $\delta_n(x) = n/\sqrt{\pi}\, e^{-n^2x^2}$, for $n = 1, 2, 5$ used to approximate the Dirac δ function when high values of n are chosen.

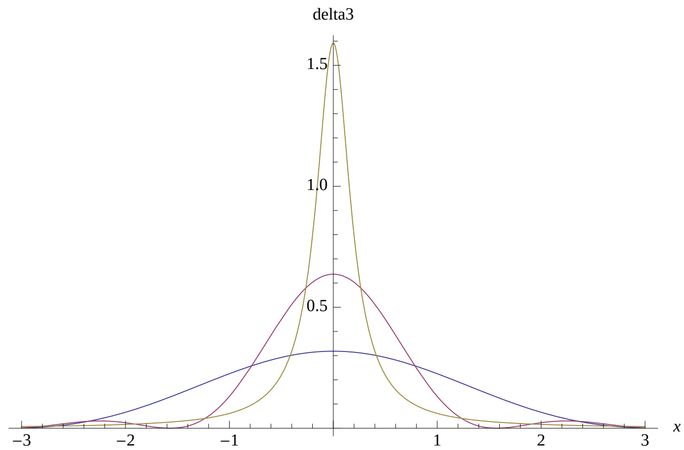

Figure 5: The sequence of functions $\delta_n(x) = \sin^2 nx/(n\pi\, x^2)$, for $n = 1, 2, 5$ used to approximate the Dirac δ function when high values of n are adopted.

$*$ These illustrations clearly indicate that once n is considerably greater than unity, the detailed form of the approximating functional sequence is actually immaterial, just the generic peaking, narrowing and area normalization character.

2.1 Parseval's Relation and Uncertainty Principle

A mathematical property of Fourier transforms that has extremely important implications for various areas of physics is **Parseval's Relation** that relates the inner products of a function and its Fourier transform: **A&W pp. 952–3**

$$\int_{-\infty}^{\infty} \Big| f(x) \Big|^2 dx \; = \; \int_{-\infty}^{\infty} \Big| g(s) \Big|^2 ds \quad . \tag{34}$$

Note that this relationship may be modified by factors like $1/2\pi$ if the normalization convention in Eqs. (27) and (28) is altered.

$*$ Observe that convergence requirements for these integrals are actually more restrictive than those for the existence of the Fourier transform.

$*$ This result can actually be extended to the convolution of two different functions.

To prove Parseval's Theorem, we make use of the integral identity for the Dirac delta function.

$$\begin{aligned}
\int_{-\infty}^{\infty} \Bigl| f(x) \Bigr|^2 dx \;&=\; \int_{-\infty}^{\infty} f(x)\, f^*(x)\, dx \\
&=\; \int_{-\infty}^{\infty} dx\, \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)\, e^{ixs}\, ds \right\} \\
&\qquad \times \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g^*(s')\, e^{-ixs'}\, ds' \right\} \\
&=\; \int_{-\infty}^{\infty} ds\, g(s) \int_{-\infty}^{\infty} ds'\, g^*(s')\, \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix(s-s')}\, dx \\
&=\; \int_{-\infty}^{\infty} ds\, g(s) \int_{-\infty}^{\infty} ds'\, g^*(s')\, \delta(s-s') \;=\; \int_{-\infty}^{\infty} \Bigl| g(s) \Bigr|^2 ds \quad .
\end{aligned} \tag{35}$$

• **Example 4:** The Fourier transform of a Gaussian is another Gaussian. To demonstrate this, we start with **A&W p. 932**

$$f(x) \;=\; \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/2\sigma^2} \quad , \tag{36}$$

of unit normalization, and a width (standard deviation) $\Delta x \sim \sigma$. Then

$$\begin{aligned}
g(s) \;&=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ixs}\, dx \\
&=\; \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2 - ixs}\, dx \\
&=\; \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2\sigma^2} \Bigl[(x+\sigma^2 is)^2 + \sigma^4 s^2 \Bigr] \right\} dx \\
&=\; \frac{1}{2\pi\sigma}\, e^{-\sigma^2 s^2/2} \int_{-\infty+i\sigma^2 s}^{\infty+i\sigma^2 s} e^{-z^2/2\sigma^2}\, dz \quad , \quad z \;=\; x + i\sigma^2 s \\
&=\; \frac{1}{\sqrt{2\pi}}\, e^{-\sigma^2 s^2/2} \quad ,
\end{aligned} \tag{37}$$

using the Residue theorem to evaluate the integral of the Gaussian by equating it to one along the real axis (there are no poles for the Gaussian). Observe that this Fourier transform has a width $\Delta s \sim 1/\sigma$ so that we deduce the correlation $\Delta x \Delta s \sim 1$ relating the widths of the Gaussian and its transform.