# 4. SERIES AND SUMMATION

Matthew Baring — Lecture Notes for PHYS 516, Fall 2022

## 1 Common Examples of Infinite Series

We begin the chapter by briefly highlighting some popular examples of finite and (mostly) infinite series. First up is the **binomial series**, whose coefficients define **Pascal's triangle**:

$$(1+x)^n \;=\; 1 + n\,x + n(n-1)\,\frac{x^2}{2!} + \ldots \;=\; \sum_{m=0}^{\infty} \frac{n!}{(n-m)!\,m!}\,x^m \quad . \tag{1}$$

For integer $n$, the series terminates at $n+1$ terms. For $n \neq 0, 1, 2, 3, 4, \ldots$, the series has infinitely many terms, but still has a valid representation within its *radius of convergence* (see below). For example, setting $n = -1$ gives the **geometric series**:

$$\frac{1}{1+x} \;=\; 1 - x + x^2 - x^3 + x^4 + \ldots \tag{2}$$

which converges for $|x| < 1$ by the ratio test (see below). This Taylor series is most useful when $|x| \ll 1$.

• Integrating the above geometric series term by term (an operation not yet demonstrated to be valid) yields the logarithmic series: **A & W, Sec. 5.6**

$$\log_e(1+x) \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \;=\; \sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{x^n}{n} \quad , \tag{3}$$

which likewise converges for $|x| < 1$.

• If we replace $x$ by $x^2$ in the geometric series, and then integrate term by term, we generate

$$\arctan x \;=\; x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \;=\; \sum_{n=0}^{\infty}(-1)^{n+1}\,\frac{x^{2n+1}}{2n+1} \quad , \qquad (4)$$

which yet again converges for $|x| < 1$.

• To move to different class of series, we offer the familiar example of the **exponential series**

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \;=\; \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad , \qquad (5)$$

which converges for all real $x$. It clearly satisfies $d(e^x)/dx = e^x$, which can be taken as the *definition* of the exponential function.

∗ Replacing $x$ by $ix$, and taking real and imaginary parts, one quickly generates the **trigonometric series**:

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \;=\; \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \quad , \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots ; = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad , \end{aligned} \qquad (6)$$

both of which also converge for all real $x$.

• Such series can be employed as a means to compute such functions, or to manipulate various sequences of operations involving such functions. Before proceeding to examples that illustrate the uses of series, we need to discuss series convergence in more depth.

• All of these series are specific examples of **Taylor series**, expansions (in these cases about $x = 0$) as series of successively higher-order derivatives.

## 2 Convergence

First, we define convergence formally. An infinite series $\sum a_n$ **converges to the sum** $S$ if **A & W, Sec. 5.1**

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} a_n\right) = S \quad , \tag{7}$$

i.e. the limit exists and equals $S$. The series **converges absolutely** if $\sum |a_n|$ converges. Observe that absolute convergence implies convergence, but not vice versa.

- **Example 1:** The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \tag{8}$$

converges to $\log_e 2$ but is not absolutely convergent, since the *harmonic series* is divergent.

- **Theorem:** If the terms $a_n$ *alternate* in sign, and if $|a_n|$ *decreases monotonically* as $n$ increases, then the sum $\sum a_n$ converges (but not necessarily absolutely).

- **Example 2:** Consider again the series

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \quad . \tag{9}$$

For $x > 0$, the signs alternate, but is $|a_n| = x^n/n$ a monotonically-decreasing function of $n$? Take

$$\frac{d}{dn}\left(\frac{x^n}{n}\right) = \frac{x^n}{n^2}\,(n\log_e x - 1) = 0 \quad \Rightarrow \quad n = \frac{1}{\log_e x} \quad . \tag{10}$$

Hence, if $0 \le x \le 1$ the derivative is *always negative*, and $|a_n|$ is monotonically decreasing with $n$, so that the series converges by the theorem.

$*$ For $x > 1$, one can always find an $n$ above which $|a_n|$ starts increasing with $n$. Accordingly, the theorem does not guarantee convergence, and the series is, in fact, divergent.

• Assessing convergence is an important protocol, since often we want to perform multiple operations on series, such as differentiation and integration. Reversing the orders of summation and other operations often can lead to tractable results. In general, this can only be applied to absolutely convergent series, which is why we want to identify domains of absolute convergence.

• **Example 3:** We illustrate the power of reversing orders of operations and summation using the series

$$f(x) \;=\; \frac{1}{1.2} + \frac{x}{2.3} + \frac{x^2}{3.4} + \dots \;=\; \sum_{k=1}^{\infty} \frac{x^{k-1}}{k(k+1)} \quad . \tag{11}$$

The series can be summed by forming $x^2 f(x)$ and then taking two derivatives, presuming such an interchange is permissible. Then

$$\frac{d^2}{dx^2}\left[x^2 f(x)\right] \;=\; \sum_{k=1}^{\infty} x^{k-1} \;\equiv\; \sum_{k=0}^{\infty} x^k \;=\; \frac{1}{1-x} \quad . \tag{12}$$

This ODE can be integrated routinely to yield a particular integral:

$$x^2 f(x) \;=\; cx + d + (1-x)\,\log_e(1-x) \quad . \tag{13}$$

This clearly must approach zero as $x \to 0$, quickly establishing that $d = 0$. Then, expanding the right hand side in a Taylor series about $x = 0$ using

$$\log_e(1-x) \;=\; -x - \frac{x^2}{2} + O(x^3) \tag{14}$$

leads to the deduction that $c = 1$ since $f(0) = 1/2$. It follows that the series can be written in closed form:

$$f(x) \;=\; \frac{1}{x} + \frac{1-x}{x^2}\,\log_e(1-x) \quad . \tag{15}$$

∗ This manipulation is viable generally on the domain $|x| < 1$ where the series is absolutely convergent. To prove such, one would truncate the series, reverse operations, and carefully deal with bounded remainders.

• To assess convergence, we need useful and powerful tests that are simple. The simplest is the **ratio test**, but it often needs refinement.

## 2.1 D'Alembert's and Cauchy's Ratio Test

The development of the ratio test is benchmarked against the geometric series, which when truncated, takes the form

**A & W, Sec. 5.2**

$$S_N \;=\; 1 + x + x^2 + \ldots + x^N \;=\; \sum_{n=0}^{N} x^n \;\equiv\; \frac{1 - x^{N+1}}{1 - x} \quad . \qquad (16)$$

The last identity can be proved by multiplying both sides by $1 - x$ . Thus,

$$\frac{1}{1 - x} \;=\; \sum_{n=0}^{N} x^n + \frac{x^{N+1}}{1 - x} \quad . \qquad (17)$$

Clearly the remainder term $x^{N+1}/(1 - x)$ in this relationship does not converge as $N \to \infty$ if $|x| > 1$ . In contrast, if $|x| < 1$ the remainder approaches zero as $N \to \infty$ , indicating convergence of the series. For $x = -1$ the series does not converge, oscillating to infinity. Evidently, we have defined a radius of convergence for this series, a concept that can be extrapolated to much broader cases:

- **D'Alembert's/Cauchy's ratio test:** if the ratio of magnitudes of successive terms of a series has a limit $r$ as $n \to \infty$ , i.e.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \;=\; r \qquad (18)$$

then the series converges (diverges) if this limit is $r < 1$ ( $r > 1$ ). If the limit (**radius of convergence**) is less than unity, then convergence is absolute.

$*$ In the geometric series example, $r = |x|$ and so the ratio test is underpinned by comparison with the geometric series.

$*$ The tricky case is when $r = 1$ , a common occurrence, for which the series may or may not converge. Such cases require more sophisticated tests.

- **Example 4:** Consider yet again the logarithmic series

$$\log_e(1 + x) \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \;=\; \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad . \qquad (19)$$

Then the radius of convergence is also $r = |x| < 1$ , which is not surprising since this series is simply obtained from the geometric one by integration.

## 2.2 The Cauchy-Maclaurin Integral Test

Another useful test is the **Cauchy-Maclaurin integral test**: a series $\sum f(n)$ converges if and only if the infinite integral

$$\int_1^\infty f(x)\,dx \tag{20}$$

converges, *provided $f(x)$ is monotonically decreasing*. This can be proven by considering the integrals and areas in Fig. 1. The left panel bounds the series from above and the right one from below. **A & W, pp. 328–9**

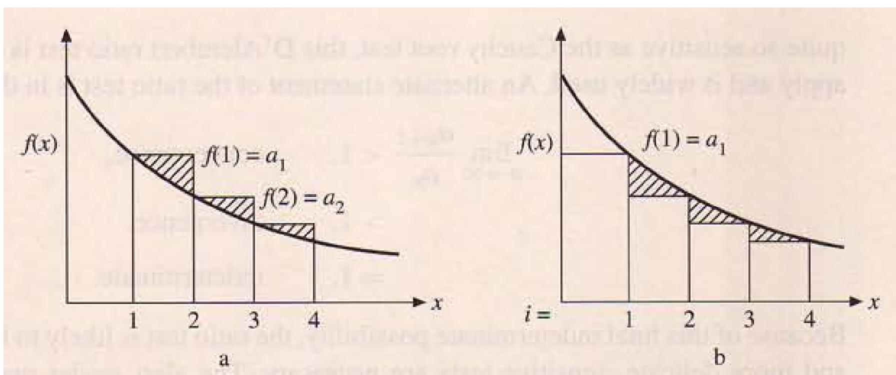



Figure 1: Graphical structure defining integral bounds for the Cauchy-Maclaurin test for series convergence. (a) At left, comparison of integral with sum blocks leading, and (b) on the right with the sum blocks lagging.

The bounds correspond to the inequalities

$$\int_1^\infty f(x)\,dx \;\leq\; \sum_{n=1}^{\infty} f(n) \;\leq\; \int_1^\infty f(x)\,dx + f(1) \quad . \tag{21}$$

Since $f(n)$ is monotonic, then it becomes clear that the existence and finiteness of the infinite integrals forces the series to converge.

• It must be emphasized that *monotonicity is required* here. Oscillatory cases can correspond to integral convergence but not series convergence. A pathological example is provided by $f(n) = \sin[(2n+1/2)\pi]/n$, which is a divergent harmonic series. Yet the integral of $f(x)$ is convergent on $[1,\infty)$.

• **Example 5:** Consider the series for the **Riemann zeta function**:

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \quad . \tag{22}$$

For this,

$$\frac{a_{n+1}}{a_n} \;=\; \left( \frac{n}{n+1} \right)^s \;=\; 1 - \frac{s}{n} + O\left( \frac{1}{n^2} \right) \;\to\; 1 \quad \text{as} \quad n \;\to\; \infty \quad , \tag{23}$$

so that the simple ratio test is inconclusive. However, $f(n) = n^{-s}$ is monotonically declining in $n$, and the integral test gives convergence for $s > 1$. Convergence is slow when $s \lesssim 2$, so an acceleration technique is then needed.

## 2.3 Gauss' Refined Ratio Test

The ratio test can be polished to address cases where $r = 1$ is the limiting ratio of terms. This is the essence of **Gauss' test**, which states that if the ratio of consecutive terms satisfies

**A & W, pp. 332–3**

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \;=\; 1 - \frac{s}{n} + \frac{B(n)}{n^2} \quad , \tag{24}$$

where $B(n)$ is bounded as $n \to \infty$, then the series converges (diverges) if this limit satisfies $s > 1$ ( $s \leq 1$ ).

• Clearly this powerful test (useful for most pathological series a physicist encounters) can be proved for $s \neq 1$ by comparison with the Riemann zeta function series. For $s = 1$, divergence is best proved by comparison with the series

$$\sum_{n=1}^{\infty} \frac{1}{n+k} \quad \Rightarrow \quad \frac{a_{n+1}}{a_n} \;\to\; 1 - \frac{1}{n} + \frac{k+1}{n^2} \quad , \tag{25}$$

whose divergence is established using the Cauchy-Maclaurin integral test.

• Example 5.2.4 in Arfken & Weber illustrates how the series solution to Legendre's equation corresponds to an $s = 1$ situation, and therefore is formally divergent if not truncated owing to integer indices $n$.

# 3 Improvement of Convergence

For efficient numerical evaluation of series, it is often important to accelerate the rate of convergence. There are several standard tools of the trade to effect such; here we will highlight **three**: Kummer's method, development of rational approximations, and Euler's transformation.

## 3.1 Kummer's Technique

The essence of **Kummer's method** is to form convenient linear combinations between the series in question, and a slowly converging series with known sums. These combinations are chosen such that the resultant series possesses a faster rate of convergence. While there is no unique choice of comparison series, Kummer opted for the convenient (i.e. broadly useful) set

**A & W, pp. 334–5**

$$\begin{aligned} \alpha_1 &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \quad , \\ \alpha_2 &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4} \quad , \\ &\vdots \\ \alpha_p &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)\dots(n+p)} = \frac{1}{p\,(p!)} \quad . \end{aligned} \tag{26}$$

The first of these can be proven by appropriate partial fractions, and grouping and relabelling of series terms

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \lim_{N\to\infty} \left\{ 1 + \sum_{n=2}^{N} \frac{1}{n} - \sum_{n=2}^{N} \frac{1}{n} \right\} = 1 \quad . \tag{27}$$

The general result can be proven by mathematical induction.

• We first apply Kummer's technique to Riemann's zeta function $\zeta(3)$, which converges at a rate similar to $\alpha_2$. Form the linear combination

$$\zeta(3) + \frac{\kappa}{4} \ = \ \sum_{n=1}^{\infty} \frac{1}{n^3} + \kappa\, \alpha_2 \quad . \tag{28}$$

This rearranges to the identity

$$\zeta(3) + \frac{\kappa}{4} \ = \ \sum_{n=1}^{\infty} \left( \frac{1}{n^3} + \frac{\kappa}{n(n+1)(n+2)} \right) \quad . \tag{29}$$

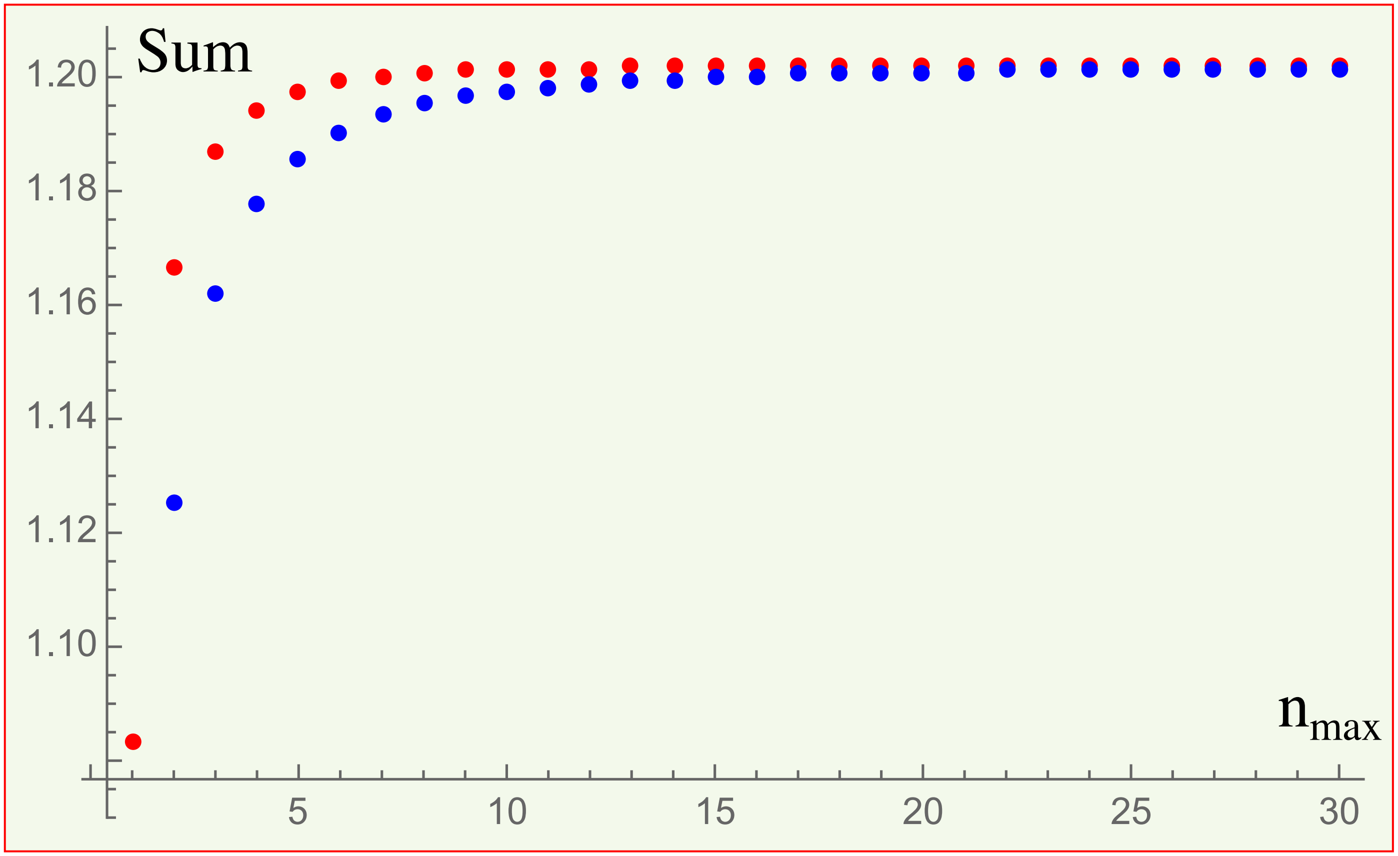
Clearly, the choice $\kappa = -1$ accelerates the convergence of the resultant series, leading to

$$\zeta(3) \ = \ \frac{1}{4} + \sum_{n=1}^{\infty} \frac{3n+2}{n^3(n+1)(n+2)} \quad , \tag{30}$$

so that convergence now goes as $1/n^4$. Using the `Mathematica` operation `Sum[f(n), {n, 1, N}]`, summing this to 10 terms gives $1.20132$ and the sum to 30 yields $1.20202$, very close to the precise value $\zeta(3) \approx 1.20206$.

**Plot:** Accelerating Riemann $\zeta(3)$ Series

# Accelerating Riemann ζ(3) Series




- Comparison of the cumulative sum $\Sigma_n 1/n^3$ for the Riemann zeta function $\zeta(3)=1.20206\ldots$ (blue dots), as a function of $n_{max}$, and the accelerated form (red dots) using Kummer's technique, which roughly halves the number of summed terms required for a given precision.