3 Linear ODEs with Non-Constant Coefficients

Now the pedagogy proceeds to generalize to non-constant coefficients, but restricts itself to second order equations to render the analytic solution techniques amenable. So, we have an inhomogeneous ODE **A & W, Sec. 9.7**

$$\frac{d^2y}{dx^2} + p(x)\,\frac{dy}{dx} + q(x)\,y \;=\; f(x) \quad . \tag{22}$$

Presume for now that we can find a **complimentary function** $y_c = c_1 y_1(x) + c_2 y_2(x)$. Replace the constants by some unknown functions $u_i(x)$ to symbolically write the **particular integral** (to be found) in the form

$$y_p(x) \;=\; u_1(x)\,y_1(x) + u_2(x)\,y_2(x) \quad . \tag{23}$$

This serves as a solution to our original ODE. This step can be taken without loss of generality, and the goal is to elicit an analytic form for $y_p(x)$. We can then write down the derivative:

$$y_p'(x) \;=\; u_1(x)\,y_1'(x) + u_2(x)\,y_2'(x) + \Big\{ u_1'(x)\,y_1(x) + u_2'(x)\,y_2(x) \Big\} \quad . \tag{24}$$

Because of our freedom in the choice of the u_i functions, we can restrict them so that the terms within the parenthesis can be set identically to zero:

$$u_1'(x)\,y_1(x) + u_2'(x)\,y_2(x) \;=\; 0 \quad . \tag{25}$$

This <u>convenient assumption</u> then leads to

$$\begin{aligned} y_p'(x) &= u_1(x)\,y_1'(x) + u_2(x)\,y_2'(x) \\ y_p''(x) &= u_1'(x)\,y_1'(x) + u_2'(x)\,y_2'(x) + u_1(x)\,y_1''(x) + u_2(x)\,y_2''(x) \quad . \end{aligned} \tag{26}$$

Insertion of these derivatives and Eq. (23) into the original ODE, Eq. (22) yields the second constraining equation for the u_i functions:

$$u_1'(x)\,y_1'(x) + u_2'(x)\,y_2'(x) \;=\; f(x) \quad . \tag{27}$$

In navigating this step, remember that y_1 and y_2 are solutions of the homogeneous ODE.

Simultaneous solution of Eqs. (25) and (27) gives

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} \;=\; \frac{1}{W(y_1,\, y_2)} \begin{pmatrix} -y_2\, f \\ y_1\, f \end{pmatrix} \quad , \quad W(y_1,\, y_2) \;\equiv\; y_1\, y_2' - y_1'\, y_2 \quad . \tag{28}$$

The function $W(y_1,\, y_2)$ is known as the **Wronskian** of the two linearly-independent complementary functions. Eq. (28) can then be simply integrated as first order ODEs in the u_i to yield

$$y_p(x) \;=\; \int^x \mathcal{G}(x,\, x')\, f(x')\, dx' \quad , \tag{29}$$

where the kernel is

$$\boxed{\mathcal{G}(x,\, x') \;\equiv\; \frac{y_1(x')\, y_2(x) - y_1(x)\, y_2(x')}{W[y_1(x'),\, y_2(x')]}} \tag{30}$$

and is entitled the **Green's function**. Mathematically, the Green's function encapsulates the information of the differential operators in the ODE, which often represent physically the force or action-at-a-distance character of the field. Accordingly, this linear problem yields a particular solution that is a sum (i.e. integral) over the *source field* $f(x)$, weighted by the *physics kernel.*

- **Example 4:** Consider the ODE (M & W p. 10)

$$\frac{d^2y}{dx^2} - \frac{2}{x^2}\, y \;=\; \frac{1}{x} \quad . \tag{31}$$

This has complementary functions $y = x^m$, $m = 2,\, -1$. Hence, the Wronskian is $W = -3$, and the particular integral assumes the form $y_p = u_1 x^2 + u_2/x$. The Green's function is then

$$\mathcal{G}(x,\, x') \;=\; \frac{x^3 - (x')^3}{3x\, x'} \quad , \tag{32}$$

and the particular integral is

$$y_p(x) \;=\; \int^x \mathcal{G}(x,\, x')\, f(x')\, dx' \;=\; -\frac{x}{2} + c_1 x^2 + \frac{c_2}{x} \quad , \tag{33}$$

Unfortunately, there is no broadly applicable procedure to find the general solution of the homogeneous ODE with non-constant coefficients, to start this process off. However, often, one solution $y_1(x)$ is easily discerned, and if it is known, **another linearly independent solution** $y_2(x)$ **can be found** as follows. Observe that the Wronskian can be differentiated thus:

$$W(x) \;\equiv\; y_1\, y_2' - y_1'\, y_2 \quad \Rightarrow \quad \frac{dW}{dx} \;=\; y_1\, y_2'' - y_1''\, y_2 \quad , \tag{34}$$

with two cancelling terms resulting. The complementary functions ($y_2(x)$ is at present unknown) satisfy the ODE:

$$y_1'' + p\, y_1' + q\, y_1 \;=\; 0 \quad \text{and} \quad y_2'' + p\, y_2' + q\, y_2 \;=\; 0 \quad . \tag{35}$$

Multiplying the first by y_2 and subtracting it from y_1 times the second yields the first order ODE

$$y_1\, y_2'' - y_1''\, y_2 + p(x) \Bigl\{ y_1\, y_2' - y_1'\, y_2 \Bigr\} \;=\; 0 \quad \text{i.e.,} \quad \frac{dW}{dx} + p(x)\, W \;=\; 0 \tag{36}$$

The solution is (by now) routine for the complementary function:

$$\boxed{W(x) \;=\; W(x_0)\, \exp \left\{ - \int_{x_0}^{x} p(x')\, dx' \right\}} \tag{37}$$

for a constant of integration x_0 . Observe that this solution for the Wronskian is independent of either y_1 and y_2 , and depends only on the coefficient $p(x)$.

- Then it is straightforward to obtain y_2 using the Wronskian:

$$\frac{d}{dx} \left(\frac{y_2}{y_1} \right) \;=\; \frac{y_1\, y_2' - y_1'\, y_2}{y_1^2} \;\equiv\; \frac{W(x)}{y_1^2} \quad \Rightarrow \quad y_2 \;=\; y_1 \int_{x_0}^{x} \frac{W(x')}{[y_1(x')]^2}\, dx' \tag{38}$$

This approach yields fruitful results. Often, as we shall see, special functions and orthogonal polynomials will be routinely identified as series solutions of particular second order ODEs. These will then seed the determination of second, linearly-independent solutions that usually have very different character, for example possessing singular points.

$*$ An example of this special function connection is the familiar ordinary Bessel function $J_n(x)$ of the first kind, that has a root at the origin, and its counterpart $Y_n(x)$ of the second kind that has a singularity at $x = 0$.

4 Power Series Solutions: Frobenius' Method

Now the ODE adventure takes us to series solutions for ODEs, a technique that is often viable, valuable and informative. These can be readily applied to high order ODEs with non-constant coefficients, usually focusing on homogeneous equations, but also amenable to particular solutions. Consider **A & W, Sec. 9.5**

$$y^{(n)} + f_{n-1}(x)\, y^{(n-1)} + \ldots + f_1(x)\, y' + f_0(x)\, y \;=\; 0 \quad . \tag{39}$$

- A point $x = x_0$ is an **ordinary point** of this ODE if $f_0,\, f_1 \ldots f_{n-1}$ are regular (i.e. analytic and single-valued) there. The general solution near such an ordinary point can be represented by a **Taylor series**:

$$y(x) \;=\; \sum_{m=0}^{\infty} c_m\, (x - x_0)^m \quad . \tag{40}$$

- A point $x = x_0$ is a **regular singular point** of this ODE if $f_0,\, f_1 \ldots f_{n-1}$ are not all regular there, but $(x - x_0)\, f_{n-1}\,, (x - x_0)^2\, f_{n-2} \ldots (x - x_0)^n\, f_0$ *are* all regular at x_0. This essentially implies that $y(x)$ must have a fixed order divergence (or pole) at x_0. The general solution near such a regular singular point can be represented by a **Frobenius series**:

$$y(x) \;=\; (x - x_0)^s \sum_{m=0}^{\infty} c_m\, (x - x_0)^m \quad , \tag{41}$$

with $c_0 \neq 0$. Note that s is not necessarily an integer.

- **Example 5:** Consider Legendre's ODE for the polynomials $P_n(x)$:

$$(1 - x^2)\, \frac{d^2 y}{dx^2} - 2x\, \frac{dy}{dx} + n(n+1)\, y \;=\; 0 \quad . \tag{42}$$

Expressed in the form of Eq. (39), this has coefficient functions of

$$f_1(x) \;=\; -\frac{2x}{1 - x^2} \quad , \quad f_0(x) \;=\; \frac{n(n+1)}{1 - x^2} \quad . \tag{43}$$

Hence, both f_0 and f_1 have simple poles at $x = \pm 1$, and these are *regular singular points* by the above definition.

$*$ Clearly, $x = 0$ is an *ordinary point* of Legendre's ODE, because f_0 and f_1 are both regular there. This then motivates the Taylor series expansion about $x = 0$:

$$y(x) \;=\; \sum_{m=0}^{\infty} c_m\, x^m \quad , \quad y'(x) \;=\; \sum_{m=0}^{\infty} m\, c_m\, x^{m-1} \quad ,$$
$$y''(x) \;=\; \sum_{m=0}^{\infty} m(m-1)\, c_m\, x^{m-2} \quad . \tag{44}$$

These are then routinely substituted into Legendre's equation to yield

$$\sum_{m=0}^{\infty} \Big[\, (1-x^2)\, m(m-1)\, c_m\, x^{m-2} - 2m\, c_m\, x^m + n(n+1)\, c_m\, x^m \,\Big] \;=\; 0 \quad . \tag{45}$$

Next, we resolve this series by rendering all terms into forms proportional to x^m, which amounts to the substitution $m \to m+2$ in just one of the $y''(x)$ contributions. This facilitates the development:

$$\sum_{m=0}^{\infty} \Big[\, (m+1)(m+2)\, c_{m+2} - m(m-1)\, c_m \;- 2m\, c_m + n(n+1)\, c_m \,\Big]\, x^m \;=\; 0 \quad . \tag{46}$$

Note, however, it comes at the price of potentially introducing a "boundary condition" or constraint on c_2, c_1 and c_0, because of the *ad hoc* addition of two extra terms at the end of the series. Fortunately, in this case, the coefficients of these extra terms are proportional to $m(m-1)$ for $m = 0, 1$, and so are identically zero. *This is not always the case.*

• Since Eq. (46) must be valid for arbitrary x on the interval $[-1,\, 1]$, each of the coefficients of x^m must be identically zero, and one arrives at the two-term recurrence relation:

$$\frac{c_{m+2}}{c_m} \;=\; \frac{m(m+1) - n(n+1)}{(m+1)\,(m+2)} \;\equiv\; \frac{(m-n)\,(m+n+1)}{(m+1)\,(m+2)} \quad . \tag{47}$$

The solutions to Eq. (42) satisfying the constraint in Eq. (47), as applied to the Taylor series, are known as **Legendre functions**.

$*$ Observe that for $m = n$ or $m = -(n+1)$, the series is truncated after a finite number of terms. Hence, if n is an integer (positive or negative), the solutions so derived are known as **Legendre polynomials** $P_n(x)$.

• The technique we have employed to ascertain series solution to Legendre's ODE is generally called **Frobenius' method**.

• If one chooses $c_1 = 0$, then $c_m = 0$ for all *odd* m, and the solution is even in x. Alternatively, if one sets $c_0 = 0$, then $c_m = 0$ for all *even* m, and the solution is odd in x, i.e. of opposite **parity**.

∗ This dichotomization delineates two *linearly independent* solutions (one even, one odd) of Legendre's ODE.

∗ The compact mathematical form for Legendre polynomials of the first kind is

$$P_n(x) \;=\; \frac{1}{2^n}\sum_{m=0}^{[n/2]} \frac{(-1)^m\,(2n-2m)!}{m!\,(n-m)!\,(n-2m)!}\; x^{n-2m} \quad , \tag{48}$$

where the notation $[n/2]$ signifies the largest integer not exceeding $n/2$.

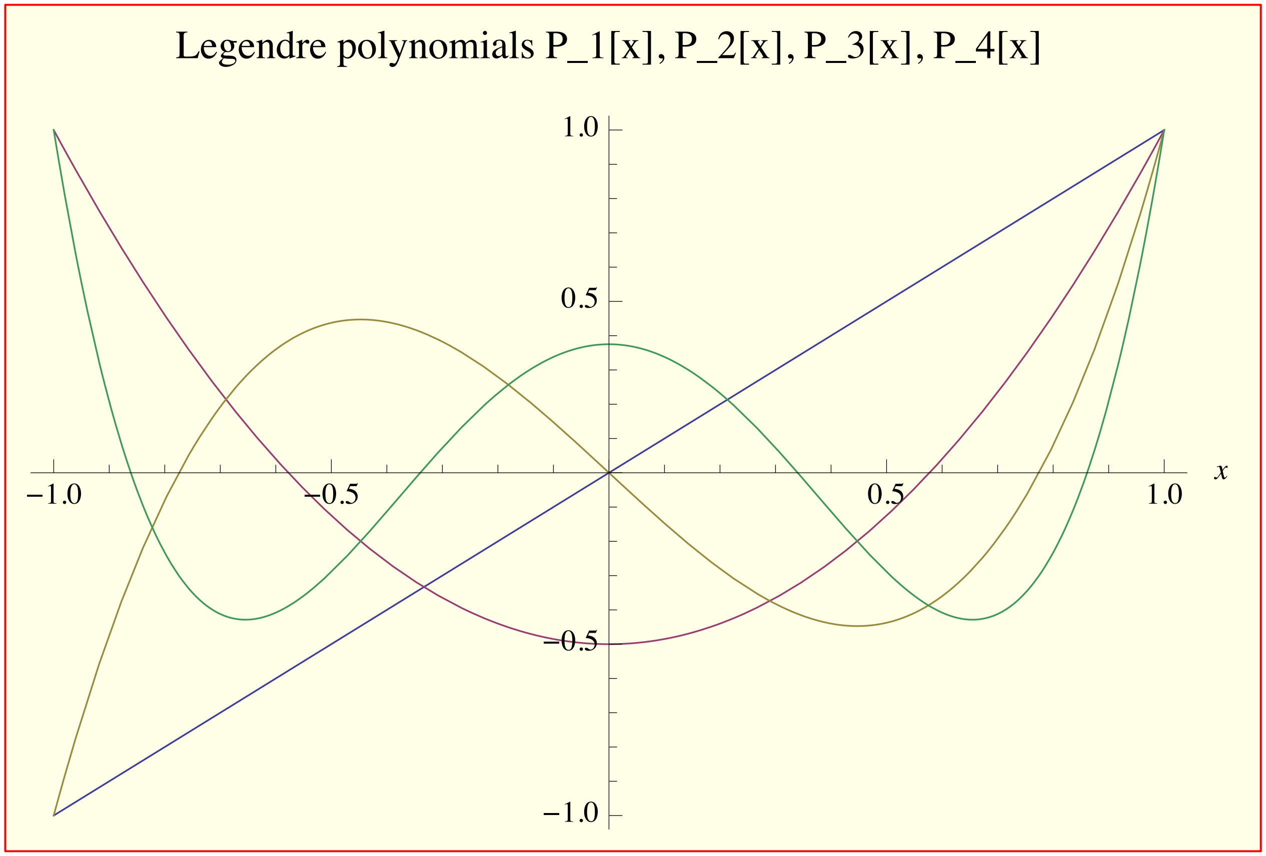
$\boxed{\textbf{Plot:}}$ Legendre Polynomials $P_n(x)$ for $n = 1, 2, 3, 4$

• Legendre's ODE usually arises in treating the θ angular part of partial differential equations with spherical symmetry (e.g. Laplace's equation, $\nabla^2\phi = 0$, or the Schrödinger equation) by the technique of separation of variables (to be encountered later). Then x represents $\cos\theta$, and physically-realistic solutions must be bounded on the interval $0 \le \theta \le \pi$, i.e. $|x| \le 1$.

• The above series solutions clearly diverge (proof to be evident in the series analysis portion of the course) if $x = \pm 1$ (the poles), unless they are truncated, i.e. n is an integer. Accordingly, the physical imposition of boundedness at the endpoints $x = \pm 1$ spawns the mathematical concept of **eigenvalues**, or the physics concept of **quantum numbers**, $n =$ integer.

∗ As we shall see, when treating partial differential equations, such physical impositions are commonplace, and yield a number of quantum numbers that often matches (but never exceeds) the dimensionality of the problem.

Legendre Polynomials $P_n(x)$



- Legendre polynomials (here for n=1,2,3,4) are bounded for all x.
- Yet, x=1 are *regular singular points*, lead to second linearly independent solutions for each n that are unbounded as $|x| \to 1$, displaying logarithmic divergences that are discerned using the Wronskian analysis.

• **Example 6:** Now, consider the Schrödinger equation for a one-dimensional harmonic oscillator:

$$\frac{d^2\psi}{dx^2} + (E - x^2)\,\psi \;=\; 0 \quad . \tag{49}$$

Here, x is a spatial coordinate, E is a constant (scaled energy), and ψ is the wavefunction. Clearly, $x = 0$ is an ordinary point of the ODE, and so we could perform a Taylor series expansion about it. However, this yields a 3-term recurrence relation, which is inconvenient.

$*$ To avoid this, the essential exponential character of the solution at large $|x|$ can be extracted. This arises for $x^2 \gg E$, for which $\psi'' - x^2\psi \approx 0$, leading to a solution $\psi \to \exp\{\pm x^2/2\}$ as $|x| \to \infty$, only one alternative of which is bounded. This motivates the substitution

$$\begin{aligned} \psi(x) \;=\; y(x)\, e^{-x^2/2} \quad , \quad \psi' \;=\; (y' - x\,y)\, e^{-x^2/2} \quad , \\ \psi'' \;=\; \bigl[y'' - 2x\,y' - (1 - x^2)\,y\bigr]\; e^{-x^2/2} \quad . \end{aligned} \tag{50}$$

The resulting ODE for y is then

$$\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + (E - 1)\,y \;=\; 0 \tag{51}$$

which can be recognized as **Hermite's ODE** by setting $E = 1 + 2n$.

• The Taylor series expansion about $x = 0$ in Eq. (44) can now be employed, along with the series manipulations that were applied to the Legendre function example. The result is

$$\sum_{m=0}^{\infty}\Bigl[\,(m+1)(m+2)\,c_{m+2} + (E - 1 - 2m)\,c_m\,\Bigr]\,x^m \;=\; 0 \quad , \tag{52}$$

yielding a 2-term recurrence relation (for $E = 1 + 2n$)

$$\frac{c_{m+2}}{c_m} \;=\; \frac{2m + 1 - E}{(m+1)\,(m+2)} \;\equiv\; \frac{2(m-n)}{(m+1)\,(m+2)} \quad . \tag{53}$$

Again, we have two linearly-independent solutions, an even series ($c_0 \neq 0$), and an odd series ($c_1 \neq 0$). Since $c_{m+2}/c_m \to 2/m$ as $m \to \infty$, each series traces that for the exponential $\exp(+x^2)$ as $x \to \infty$, rendering the solutions for ψ divergent there, *unless the series terminate*. This happens only when n is an integer. The resulting finite power series are **Hermite polynomials** $H_n(x)$. Thus, requiring a bounded solution mandates

$$E \;=\; E_n \;\equiv\; 1 + 2n \tag{54}$$

for the *eigenvalues*, and

$$\psi(x) \;=\; \psi_n(x) \;\equiv\; H_n(x)\, e^{-x^2/2} \tag{55}$$

for the *eigenfunctions*.

• **Example 7:** Finally, we will explore solutions to Bessel's ODE for the familiar ordinary Bessel functions $J_n(x)$:

$$x^2 \frac{d^2y}{dx^2} + x\,\frac{dy}{dx} + (x^2 - n^2)\, y \;=\; 0 \quad . \tag{56}$$

This naturally arises as the radial part ($x \to \rho$) of the solutions to PDEs in cylindrical $(\rho,\, \phi,\, z)$ coordinates, for example $\nabla^2\phi = 0$ in electrostatics.

• Here, $x = 0$ is a *regular, singular point*. Thus, the solution can be expressed as a **Frobenius series**:

$$\begin{gathered} y(x) \;=\; x^s \sum_{m=0}^{\infty} c_m\, x^m \quad , \quad y'(x) \;=\; x^s \sum_{m=0}^{\infty} (m+s)\, c_m\, x^{m-1} \quad , \\ y''(x) \;\;=\;\; x^s \sum_{m=0}^{\infty} (m+s)\,(m+s-1)\, c_m\, x^{m-2} \quad . \end{gathered} \tag{57}$$

These are then routinely substituted into Bessel's equation to yield

$$\sum_{m=0}^{\infty} \Big[\, (m+s)(m+s \cancel{-1}) + \cancel{(m+s)} + (x^2 - n^2) \Big]\, c_m\, x^{m+s} \;=\; 0 \quad . \tag{58}$$

Again, one resolves this series by rendering all terms into forms proportional to the same power of x, leading to a slightly more involved situation than that for Legendre's ODE:

$$\begin{aligned} &(s^2 - n^2)\, c_0\, x^s + \left[(s+1)^2 - n^2\right]\, c_1\, x^{s+1} \\ &\quad + \sum_{m=0}^{\infty} \left\{ \left[(m+s+2)^2 - n^2\right] c_{m+2} + c_m \right\} x^{m+s+2} \;=\; 0 \quad . \end{aligned} \tag{59}$$

Accordingly, we have additional constraints. First, for $c_0 \neq 0$, the coefficient of x^s must be non-zero, yielding an **indicial equation**:

$$s \;=\; \pm n \quad . \tag{60}$$

Similarly, the coefficient of x^{s+1} must be zero, yielding either $c_1 = 0$, *or* $s + 1 = \pm n$. The former identity must apply, since the latter is clearly incompatible with the indicial equation. Hence, $c_1 = 0$, and the operating recurrence relation is

$$\frac{c_{m+2}}{c_m} \;=\; -\frac{1}{(m+s+2)^2 - s^2} \;=\; -\frac{1}{(m+2)\,(m+2+2s)} \quad . \tag{61}$$

$*$ Observe that the $s + 1 = \pm n$ indicial equation (i.e. $c_0 = 0$) actually generates the same overall solution, just with a summation index relabelling.

• For n not an integer, the two linearly-independent solutions are given by $s = \pm|n|$. This case is often not realized on physical grounds, because n frequently represents the wavenumber of the azimuthal dependence $e^{i\,n\phi}$, which must be an integer for a single-valued function.

• For n being an integer, the series is still infinite, and provides a valid solution for $s = +|n|$ that is known as the familiar **ordinary Bessel function** of the first kind:

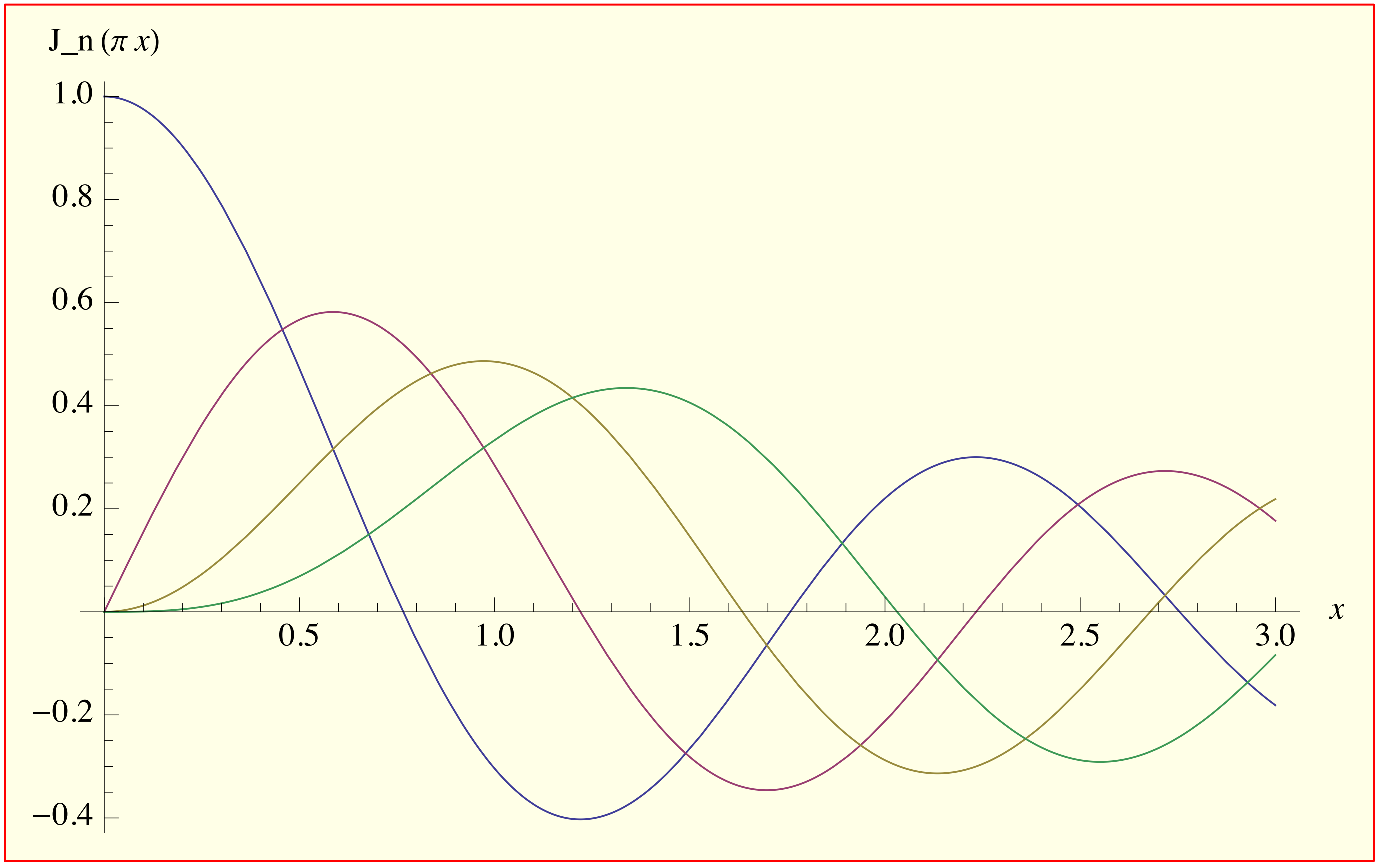
$$J_n(x) \;=\; \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(n+j)!} \left(\frac{x}{2}\right)^{n+2j} \quad , \quad n \;=\; 0,\, 1,\, 2,\, \ldots \tag{62}$$

Here we have set $2j = m + 2$.

Plot: Bessel functions $J_n(\pi x)$ for $n = 1, 2, 3, 4$

In contrast, the $s = -|n|$ series blows up, since $c_m \to \infty$ as $m \to -2s$. Hence, for integer n, a second linearly-independent solution (the $Y_n(x)$ Bessel function) must be obtained by alternative, more involved means — *for example, using Wronskian analysis.*

Bessel Functions $J_n(\pi x)$



- Ordinary Bessel functions (here for n=1,2,3,4) are bounded for all x.
- As x=0 is a *regular singular point*, the second linearly independent solutions for each n are unbounded as $|x| \rightarrow 0$, displaying logarithmic or power-law divergences that are discerned using the Wronskian analysis.