3. ORDINARY DIFFERENTIAL EQUATIONS

Matthew Baring — Lecture Notes for PHYS 516, Fall 2022

Ordinary differential equations (ODEs) arise in a multitude of contexts, often involving equations of motion in physics. They arise naturally when separating variables in partial differential equations. **A & W, Sec. 9.2**

• The **order** of an ODE is the order of the highest derivative in it.

• We focus here primarily on **linear** ODEs, where the terms involves *the function and its derivatives appearing only to the first power.*

• Non-linear ODEs are generally difficult or impossible to solve analytically. However, isolated examples are amenable to solution, particularly if they possess exact differentials that are separable in the function and its variable.

∗ An example of a non-linear ODE is the equation of motion of a ball in flight through air, where the momentum rate of change term $\propto dv/dt$ has to respond to a frictional (air-resistance) term that to leading-order scales usually as $\propto v^3$.

• **Example 1:** Consider the non-linear ODE (adapted from M/W p. 4)

$$\frac{dy}{dx} \; = \; \frac{y}{x + 2\sqrt{xy}} \quad . \tag{1}$$

The substitution $y = v\,x$ facilitates the extraction of factors on the R.H.S.:

$$\frac{dy}{dx} \;\equiv\; x\,\frac{dv}{dx} + v \;=\; \frac{v}{1 + 2\sqrt{v}} \quad\Rightarrow\quad x\,\frac{dv}{dx} \;=\; -\frac{2v^{3/2}}{1 + 2\sqrt{v}} \quad . \tag{2}$$

This is now routinely integrable. One quickly arrives at

$$
\begin{aligned}
-\left(\frac{1}{v}+\frac{1}{2v^{3/2}}\right)\, dv \;&=\; \frac{dx}{x} \quad \Rightarrow \quad -\log_e v + \frac{1}{\sqrt{v}} \;=\; \log_e x + c \\
\Rightarrow \quad \frac{1}{\sqrt{v}} \;&=\; \log_e vx + c \quad \Rightarrow \quad x \;=\; y\,\left(\log_e y + c\right)^2
\end{aligned}
\tag{3}
$$

for an integration constant c. Differentiating this solution routinely yields the original ODE in Eq. (1). Observe that the solution does not express y explicitly in terms of x, i.e. it must be implicitly derived via numerical root-solving techniques.

1 Linear, First Order ODEs

The most general first-order ODE assumes the form

A & W, pp. 547-8

$$
\frac{dy}{dx} + q(x)\, y \;=\; f(x) \quad . \tag{4}
$$

It has a single solution with one constant of integration, which may be determined by one (and only one) boundary condition on *either* $y(x_0)$ or $y'(x_0)$. To find it, we multiply both sides by the **integrating factor**

$$
\lambda(x) \;=\; \exp\left\{\int_{x_0}^{x} q(x')\, dx'\right\} \quad . \tag{5}
$$

This yields an exact differential on the L.H.S. of the ODE:

$$
\frac{d}{dx}\Big(\lambda(x)\, y\Big) \;=\; \lambda(x)\, f(x) \quad . \tag{6}
$$

The homogeneous equation, $f = 0$, yields a **complementary function**

$$
y_c(x) \;=\; \frac{y(x_0)}{\lambda(x)} \;=\; y(x_0)\, \exp\left\{-\int_{x_0}^{x} q(x')\, dx'\right\} \tag{7}
$$

Here x_0 serves as a constant of integration. The complete solution is

$$
y(x) \;=\; y_c(x) + \frac{1}{\lambda(x)} \int_{x_0}^{x} \lambda(x')\, f(x')\, dx' \quad , \tag{8}
$$

which can be rewritten in the form

$$\boxed{y(x) \;=\; y_c(x) + \int_{x_0}^{x} f(x')\, \exp\left\{ - \int_{x'}^{x} q(x'')\, dx'' \right\} dx' \quad .} \tag{9}$$

- **Example 2:** Consider the ODE

$$\frac{dy}{dx} + \frac{y}{x} \;=\; \frac{1}{x^2} \quad . \tag{10}$$

The solution in Eq. (9) can then be simply determined using $q(x) = 1/x$ and $f(x) = 1/x^2$:

$$y \;=\; \frac{\log_e x + c}{x} \quad . \tag{11}$$

2 Linear ODEs with Constant Coefficients

- Start our ODE journey simply: linear and constant coefficients.

- Linear ODEs of higher n^{th} order can be routinely solved if their coefficients are constant, i.e. independent of x . The general form of such an equation is **M & W, Sec. 1-1**

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \ldots + a_1\, y' + a_0\, y \;=\; f(x) \quad , \tag{12}$$

and it *has* n *linearly independent solutions* (forming a basis in a vector space) with n constants of integration that are determined by n (and only n) boundary conditions on $y, y', \ldots y^{(n)}$ fixed at some value $x = x_0$.

- Setting $f(x) = 0$ results in the **homogeneous equation**, whose solution is called the **complementary function** (also applicable to equations with non-constant coefficients).

- The general solution of the *inhomogeneous equation* equals the general solution of the homogeneous equation plus *any* solution of the inhomogeneous equation (which is called the **particular integral**). This defines the solution protocol that is broadly adopted.

• To solve the homogeneous equation, try the *ansatz* (the trial solution) $y = e^{mx}$ for complex m. This is motivated by it obviously being a solution to first order ODEs of this form,

$$a_1\, y' + a_0\, y \;=\; 0 \quad\Rightarrow\quad y \;\propto\; \exp\biggl\{\frac{a_0}{a_1}\, x\biggr\} \quad , \tag{13}$$

and its general facility in factoring out the exponential, with a residual n^{th} order albegraic equation in m being the result:

$$a_n\, m^n + a_{n-1}\, m^{n-1} + \ldots + a_1\, m + a_0 \;=\; 0 \quad . \tag{14}$$

This is termed the **characteristic equation**; it can be solved routinely to generate a set of **eigenvalues** $\{m_i\}$ that are in general complex.

• If all the m_i are distinct, then the **eigenvector** set $\{e^{m_i x}\}$ constitutes n linearly independent solutions (i.e., *a basis*), and the general solution $y_c(x)$ to the homogeneous equation is a linear combination of these:

$$y_c(x) \;=\; \sum_{i=1}^{n} c_i\, e^{m_i x} \quad . \tag{15}$$

Observe that $\mathrm{Re}\,(m_i) \neq 0 \Rightarrow$ an *exponential solution*, and $\mathrm{Im}\,(m_i) \neq 0 \Rightarrow$ signifies *oscillatory character* in the solution; for complex m_i, the solution exhibits both (a case exemplified by damped wave propagation in material media; and also LRC circuits).

• If two roots are equal, one can deduce the extra linearly-independent solution via an "adiabatic" perturbation of one or more of the ODE's a_j coefficients, i.e. send $a_j \to a_j + \epsilon$. Presuming that $m_1 = m_2$ are the offending roots, then we can form the limiting case of a linear combination of solutions from the perturbed ODE:

$$\lim_{m_2\to m_1} \frac{e^{m_2 x} - e^{m_1 x}}{m_2 - m_1} \;=\; \frac{d}{dm}\, e^{mx}\biggr|_{m=m_1} \;=\; x\, e^{m_1 x} \tag{16}$$

provides the additional linearly independent solution.

$*$ By extension, if there are three equal roots to the characteristic equation, then $x\, e^{m_1 x}$ and $x^2\, e^{m_1 x}$ are the additional solutions, etc.

• To find a **particular integral**, first check to see if $f(x) \sim x^n e^{\alpha x}$ is a solution of the ODE, where n is an integer and α is a complex constant. If so, then the particular integral is a linear combination of the finite number of linearly independent derivatives of $f(x)$, with coefficients determined by substitution into the ODE (**method of undetermined coefficients**).

• In general, this is not the case, so we factorize the ODE in operator form:

$$(D - m_1)(D - m_2) \ldots (D - m_n)\, y \;=\; f(x) \quad , \tag{17}$$

where $D \equiv d/dx$ is the derivative operator, and the m_i are the eigenvalues. Then one sequentially solves a number of elementary first order ODEs:

$$(D - m_1)\, y_1 \;=\; f(x) \;\;, \quad (D - m_2)\, y_2 \;=\; y_1(x) \ldots (D - m_n)\, y_n \;=\; y_{n-1}(x) \;\;, \tag{18}$$

so that $y_{\mathrm{P}} = y_n(x)$ is the particular integral. This is a routine technique.

• **Example 3:** Consider the ODE

$$\frac{d^2y}{dx^2} + k^2 y \;\equiv\; (D + i\,k)(D - i\,k)\, y \;=\; e^{-x} \quad . \tag{19}$$

The homogeneous equation yields the linearly-independent solutions $e^{\pm i\,kx}$, which form the integrating factors for the factored ODEs:

$$\begin{aligned}
(D + i\,k)\, y_1 \;=\; e^{-x} \quad &\Rightarrow \quad D\left(e^{i\,kx}\, y_1\right) \;=\; e^{i\,kx}\, e^{-x} \\
&\Rightarrow \quad y_1 \;=\; c_1\, e^{-i\,kx} - \frac{e^{-x}}{1 - i\,k} \quad , \\
(D - i\,k)\, y_2 \;=\; y_1 \quad &\Rightarrow \quad D\left(e^{-i\,kx}\, y_2\right) \;=\; c_1\, e^{-2i\,kx} - \frac{e^{-x(1+i\,k)}}{1 - i\,k} \\
&\Rightarrow \quad y_2 \;=\; -\frac{c_1}{2i\,k}\, e^{-i\,kx} + c_2\, e^{i\,kx} + \frac{e^{-x}}{1 + k^2} \quad ,
\end{aligned} \tag{20}$$

so that in terms of real variables, the solution is

$$y \;=\; a\, \cos kx + b\, \sin kx + \frac{e^{-x}}{1 + k^2} \tag{21}$$

for constants a and b that are expressible in terms of c_1 and c_2.

3 Linear ODEs with Non-Constant Coefficients

Now the pedagogy proceeds to generalize to non-constant coefficients, but restricts itself to second order equations to render the analytic solution techniques amenable. So, we have an inhomogeneous ODE **A & W, Sec. 9.7**

$$\frac{d^2y}{dx^2} + p(x)\,\frac{dy}{dx} + q(x)\,y \;=\; f(x) \quad . \tag{22}$$

Presume for now that we can find a **complimentary function** $y_c = c_1 y_1(x) + c_2 y_2(x)$. Replace the constants by some unknown functions $u_i(x)$ to symbolically write the **particular integral** (to be found) in the form

$$y_p(x) \;=\; u_1(x)\,y_1(x) + u_2(x)\,y_2(x) \quad . \tag{23}$$

This serves as a solution to our original ODE. This step can be taken without loss of generality, and the goal is to elicit an analytic form for $y_p(x)$. We can then write down the derivative:

$$y_p'(x) \;=\; u_1(x)\,y_1'(x) + u_2(x)\,y_2'(x) + \Big\{ u_1'(x)\,y_1(x) + u_2'(x)\,y_2(x) \Big\} \quad . \tag{24}$$

because of our freedom in the choice of the u_i functions, we can restrict them so that the terms within the parenthesis can be set identically to zero:

$$u_1'(x)\,y_1(x) + u_2'(x)\,y_2(x) \;=\; 0 \quad . \tag{25}$$

This convenient assumption then leads to

$$\begin{aligned} y_p'(x) &= u_1(x)\,y_1'(x) + u_2(x)\,y_2'(x) \\ y_p''(x) &= u_1'(x)\,y_1'(x) + u_2'(x)\,y_2'(x) + u_1(x)\,y_1''(x) + u_2(x)\,y_2''(x) \quad . \end{aligned} \tag{26}$$

Insertion of these derivatives and Eq. (23) into the original ODE, Eq. (22) yields the second constraining equation for the u_i functions:

$$u_1'(x)\,y_1'(x) + u_2'(x)\,y_2'(x) \;=\; f(x) \quad . \tag{27}$$

In navigating this step, remember that y_1 and y_2 are solutions of the homogeneous ODE.

Simultaneous solution of Eqs. (25) and (27) gives

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{W(y_1,\, y_2)} \begin{pmatrix} -y_2\, f \\ y_1\, f \end{pmatrix} \quad , \quad W(y_1,\, y_2) \equiv y_1\, y_2' - y_1'\, y_2 \quad . \tag{28}$$

The function $W(y_1,\, y_2)$ is known as the **Wronskian** of the two linearly-independent complementary functions. Eq. (28) can then be simply integrated as first order ODEs in the u_i to yield

$$y_p(x) = \int^x \mathcal{G}(x,\, x')\, f(x')\, dx' \quad , \tag{29}$$

where the kernel is

$$\boxed{\mathcal{G}(x,\, x') \equiv \frac{y_1(x')\, y_2(x) - y_1(x)\, y_2(x')}{W[y_1(x'),\, y_2(x')]}} \tag{30}$$

and is entitled the **Green's function**. Mathematically, the Green's function encapsulates the information of the differential operators in the ODE, which often represent physically the force or action-at-a-distance character of the field. Accordingly, this linear problem yields a particular solution that is a sum (i.e. integral) over the *source field* $f(x)$, weighted by the *physics kernel*.