3 Differentiation

Now we proceed to extend Newton's theory of differential calculus to the complex plane, a task that was extensively researched by Cauchy. A function $w = f(z)$ is defined to be **analytic** at z if it has a *derivative* there in the Newtonian sense, i.e. if **A & W, Sec. 6.2**

$$f'(z) \equiv \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{13}$$

exists, *and is independent of the path by which* $\Delta z \to 0$. This provides a robust definition of the derivative in the complex plane.

3.1 The Cauchy-Riemann Relations

The path-independence of the derivative is a crucial feature, and yields an attractive and powerful constraint on the functions $u(x, y)$ and $v(x, y)$.

Plot: General Derivation of Cauchy-Riemann Relations

For example, we can evaluate the limit in each of the x and y directions:

$$\begin{aligned} \frac{df}{dz} &= \frac{d(u + i\,v)}{d(x + i\,y)} \\ &= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \quad \text{if} \quad dy = 0 \quad , \\ &= \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y} \quad \text{if} \quad dx = 0 \quad . \end{aligned} \tag{14}$$

These must be the same, for an analytic function. Hence, equating real and imaginary parts gives the **Cauchy-Riemann relations**:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad .} \tag{15}$$

These are necessary and sufficient (not proven) for $f(z) = u(x, y) + i\,v(x, y)$ to be analytic. If $f(z)$ is analytic everywhere it is called **entire**.

Derivation of the Cauchy-Riemann Relations

The full derivation of the Cauchy-Riemann relations accommodates any paths $y = y(x)$ about a point z:

$$\frac{df}{dz} = \frac{d(u + i\,v)}{d(x + i\,y)} = \frac{(du + idv)\,(dx - idy)}{dx^2 + dy^2}$$

$$= \frac{\partial u}{\partial x}\,\frac{1}{1 + [y'(x)]^2} + \frac{\partial v}{\partial y}\,\frac{[y'(x)]^2}{1 + [y'(x)]^2}$$

$$+ i\,\frac{\partial v}{\partial x}\,\frac{1}{1 + [y'(x)]^2} - i\,\frac{\partial u}{\partial y}\,\frac{[y'(x)]^2}{1 + [y'(x)]^2} \quad .$$

For arbitrary $y(x)$ and therefore also its derivative $y'(x)$, this can only be of fixed value if both

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

are satisfied. This completes the general derivation.

• **Example 2:** Consider A/W problem 6.2.1. For an analytic function,

$$\begin{aligned}
\nabla^2 u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = 0 \\
\nabla^2 v &= \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 0 \quad ,
\end{aligned} \tag{16}$$

i.e. u and v satisfy Laplace's equation, and are called **harmonic functions**. Furthermore, another simple consequence of the C-R relations is

$$0 = -\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \equiv \vec{\nabla} u \cdot \vec{\nabla} v \quad , \tag{17}$$

so that the vector gradients of u and v in the (x, y) plane are orthogonal:

$$\vec{\nabla} u \cdot \vec{\nabla} v = 0 \quad \text{for} \quad \vec{\nabla} g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) \quad . \tag{18}$$

Accordingly, u and v are termed **conjugate functions**, and their contour plots in the (x, y) plane are everywhere orthogonal.

Plot: Contour plots of u and v for $w(z) = 1/z$

• N.B. The harmonic and conjugate nature of analytic functions renders them ideal for representing *2D potentials* such as those of the electrostatic or magnetostatic variety, or velocity streamlines in fluid flow theory.

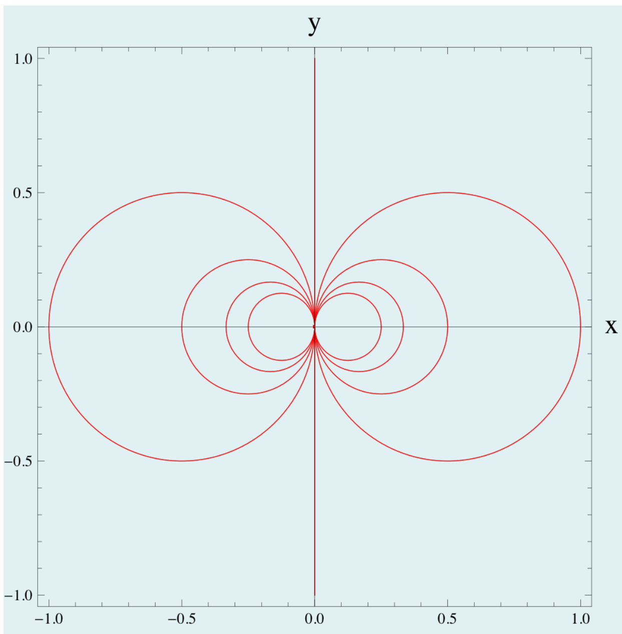
• **Example 3:** For the complex conjugation function $f(z) = z^*$, $u(x, y) = x$ and $v(x, y) = -y$ so that

$$\frac{\partial u}{\partial x} = 1 \quad , \quad \frac{\partial v}{\partial y} = -1 \quad , \tag{19}$$

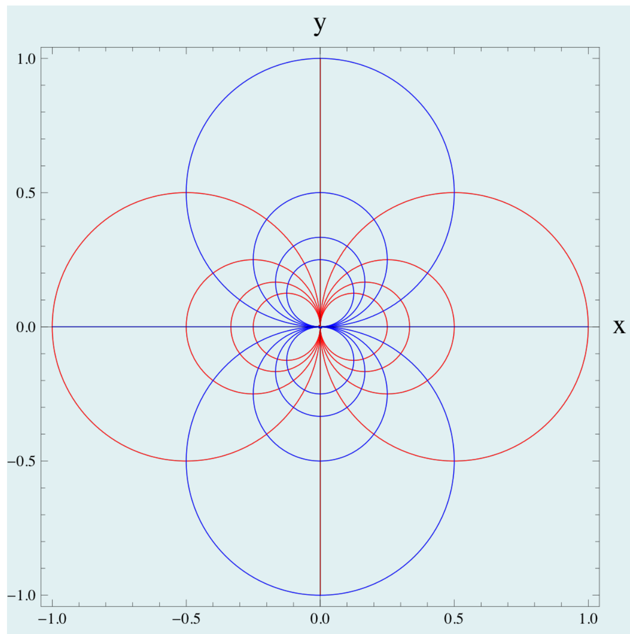
clearly implying that $f(z)$ is not analytic for *any* z; this is not surprising since it is a *reflection* mapping.

Conjugate u,v Functions for w(z) = 1/z

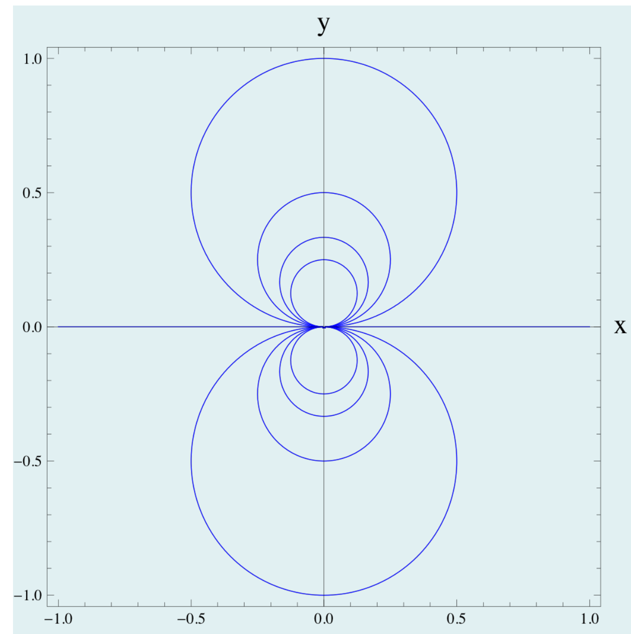
$u = \mathrm{Re}(1/z) = x/(x^2+y^2)$



(u,v) combined



$v = \mathrm{Im}(1/z) = -y/(x^2+y^2)$



- Plots were obtained with Mathematica using coding such as:

```
ContourPlot[ x/(x^2 + y^2), {x, -1, 1}, {y, -1, 1},  ContourShading -> False,
PlotPoints -> 60, ContourStyle -> { Red, Red, Red, Red, Red}, Axes -> True,
AxesLabel -> {StyleForm["x", FontSize -> 16], StyleForm["y", FontSize -> 16]}]
```

4 Integration

Integration in the complex plane serves as a rich and powerful extension of Newton's theory for the real line. If C is a piecewise continuous curve (**contour**) in the complex plane, we can define the **contour integral** **A & W, Sec. 6.3**

$$\begin{aligned} \int_C f(z)\,dz &= \int_C (u + i\,v)\,(dx + i\,dy) \\ &= \int_C (u\,dx - v\,dy) + i \int_C (v\,dx + u\,dy) \quad . \end{aligned} \tag{20}$$

More formally, the integral may be defined along the lines of Newton's theory for real integrals:

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j)\,[z_j - z_{j-1}] \tag{21}$$

for $|z_j - z_{j-1}| \to 0$ as $n \to \infty$, and the point ζ_j lying somewhere along the curve C between z_{j-1} and z_j.

4.1 The Cauchy Integral Theorem

If C is a *closed* curve and $f(z)$ is *analytic within its enclosed area*, then without proof, we state the **Cauchy Integral Theorem**:

$$\boxed{\oint_C f(z)\,dz = 0 \quad .} \tag{22}$$

It can be proven from the Cauchy-Riemann conditions, when $f'(z)$ is continuous, using Stoke's Theorem to convert the line integrals to surface ones. Alternatively, the **Goursat proof** is structured around the more formal definition of integral/differential calculus, and does not require continuity (as opposed to existence) of the first derivatives — an unecessary restriction. **A & W, pp. 420-2**

[*Reading Assignment: A/W pp. 420–422: Stoke's Theorem and Goursat Proofs of the Cauchy Integral Theorem*]

• **Example 4:** For the function $f(z) = (z-a)^{-1}$, the derivative $df/dz = -1/(z-a)^2$ exists except at the singularity $z = a$. Hence the contour integral in Eq. (22) is zero if C *does not* enclose $z = a$.

If, however, C encloses $z = a$, the Cauchy Integral Theorem cannot be applied, and the integral must be evaluated explicitly. If $z = a$ does not lie on C, it is always possible to find a circular contour C_r of radius r that is centered on $z = a$ but within C.

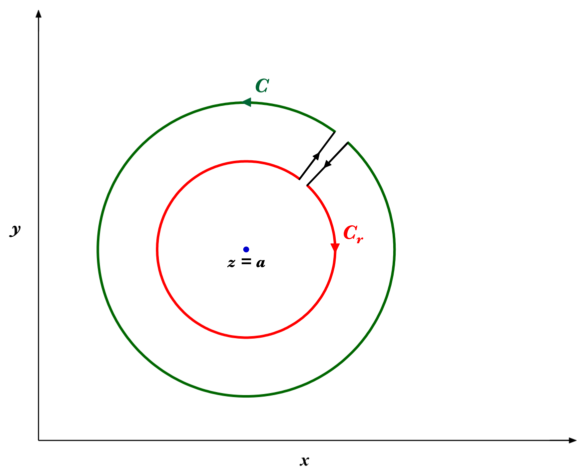


Figure 1: Contour geometry around $z = a$ for the Cauchy Integral Theorem.

By constructing a thin filamentary connection between contours C and C_r, we can apply the Cauchy Integral Theorem to establish that

$$\oint_C \frac{dz}{z-a} + \oint_{C_r} \frac{dz}{z-a} \;=\; 0 \tag{23}$$

in the limit that the filament is infinitesimally thin. Then, setting

$$z - a \;=\; r\,e^{i\,\theta} \quad \Rightarrow \quad dz \;=\; i\,r\,e^{i\,\theta}\,d\theta \quad , \tag{24}$$

the integrals can be evaluated:

$$\oint_C \frac{dz}{z-a} \;=\; -\oint_{C_r} \frac{dz}{z-a} \;=\; \int_0^{2\pi} \frac{i\,r\,e^{i\,\theta}}{r\,e^{i\,\theta}}\,d\theta \;=\; 2\pi i \quad . \tag{25}$$

• *Convention:* Contours integrate in a counterclockwise direction to generate a positive sense to the integral.

4.2 The Cauchy Integral Formula

The integration in Example 4 becomes a central foundation for the subsequent establishment of the **Cauchy Integral Formula.** If $f(z)$ is *analytic within and on* C (e.g., e^{-z}, z^{α}, $J_n(z)$, $\sin z$), and $z = z_0$ lies within, then **A & W, Sec. 6.4**

$$\boxed{f(z_0) \;=\; \frac{1}{2\pi i} \oint_C \frac{f(\zeta)\, d\zeta}{\zeta - z_0} \quad .} \tag{26}$$

The import of this theorem is that *the values of an analytic function on a closed curve determine the function everywhere within that curve.* This is a powerful property of complex analysis. Again, observe the similarity to a 2-D potential problem.

• To prove the Formula, we construct a circular contour C_ε within C as in Example 4, with C_ε of radius ε and centered on z_0, and then attach a thin filamentary connection between the two contours.

Plot: Geometry of C and C_ε contours around $z = z_0$

Then for C and C_ε being <u>*both counterclockwise*</u> contours, we have

$$\oint_C \frac{f(\zeta)}{\zeta - z_0}\, d\zeta - \oint_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta \;=\; 0 \quad . \tag{27}$$

Then, setting

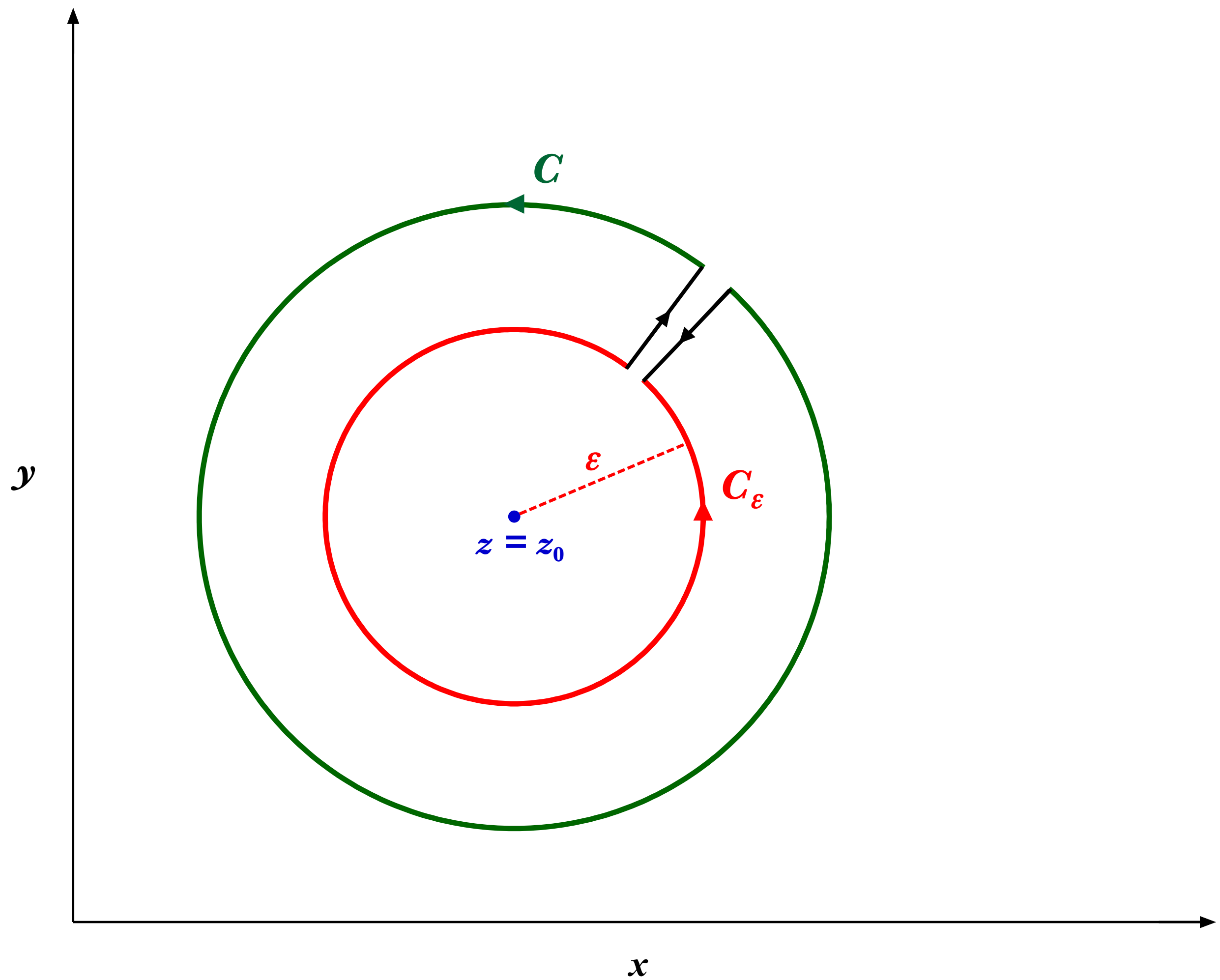
$$\zeta - z_0 \;=\; \varepsilon\, e^{i\theta} \quad \Rightarrow \quad d\zeta \;=\; i\,\varepsilon\, e^{i\theta}\, d\theta \quad , \tag{28}$$

the integrals can be evaluated by eventually taking the limit $\varepsilon \to 0$:

$$\begin{aligned} \oint_C \frac{f(\zeta)}{\zeta - z_0}\, d\zeta \;&=\; \oint_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta \;=\; \oint_{C_\varepsilon} \frac{f\left(z_0 + \varepsilon\, e^{i\theta}\right)}{\varepsilon\, e^{i\theta}}\, i\,\varepsilon\, e^{i\theta}\, d\theta \\ &\longrightarrow\; i\, f(z_0) \oint_{C_\varepsilon} d\theta \;=\; 2\pi i\, f(z_0) \quad . \end{aligned} \tag{29}$$

The Cauchy Integral Formula then follows.

Contour Geometry for Cauchy Integral Formula



5 Taylor and Laurent Series

Having established the Cauchy Integral Formula, it is now possible to discern how *derivatives of analytic functions can be expressed in contour integral form.* The first derivative can be written as follows **A & W, Sec. 6.6**

$$\begin{aligned} f'(z) &= \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \\ &= \lim_{\delta z \to 0} \frac{1}{2\pi i\, \delta z} \oint_C \left(\frac{1}{\zeta - z - \delta z} - \frac{1}{\zeta - z} \right) f(\zeta)\, d\zeta \\ &= \frac{1}{2\pi i} \oint_C f(\zeta) \frac{d}{dz} \left(\frac{1}{\zeta - z} \right) d\zeta \\ &= \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \quad , \end{aligned} \tag{30}$$

with C being any contour surrounding z, within which $f(z)$ is analytic. Likewise, using *mathematical induction* as a method of proof, this result is readily generalized to arbitrary higher order derivatives:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{n+1}} \quad . \tag{31}$$

Hence, the analytic nature of $f(z)$ *guarantees the existence of all derivatives.*

• This sets up the possibility of expressing the function $f(z)$ as a **Taylor series**. Every function that is analytic at and around $z = z_0$ possesses a unique power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad , \quad c_n = \frac{f^{(n)}(z_0)}{n!} \equiv \frac{1}{2\pi i} \oint_C \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}} \quad , \tag{32}$$

that converges within a **circle of convergence** that extends out to the nearest singularity of $f(z)$. The proof is left as a reading assignment.

[*Reading Assignment: A/W pp. 430-1: Proof of existence of a Taylor Series*]