1. COMPLEX ANALYSIS

Matthew Baring — Lecture Notes for PHYS 516, Fall 2022

1 Uses of Complex Variables

• Uses of complex variables include (but are not limited to) the following diverse tasks:

∗ manipulation of series and products

∗ solution space for homogenous linear ODEs of arbitrary order

∗ extension of the validity of solutions to second order ODEs to larger domains (**analytic continuation**)

∗ tools for solving 2-D PDEs such as Laplace's equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \; = \; 0 \tag{1}$$

∗ evaluation of definite integrals

∗ inversion of integral transforms

∗ connection of physically-related quantities under one formalism — e.g. **dispersion theory**

∗ combinatorial manipulation

2 Functions of a Complex Variable

A **complex number** has the form $z = x + iy$ where x and y are real, and $i = \sqrt{-1}$. Here, $x = \mathrm{Re}(z)$ is called the *real part*, and $y = \mathrm{Im}(z)$ is called the *imaginary part*. This defines the standard Cartesian coordinate representation. **A & W, Sec. 6.1**

- Addition of complex numbers works just as for real numbers:

$$z_1 \pm z_2 \;=\; (x_1 \pm x_2) + i\,(y_1 \pm y_2) \quad . \tag{2}$$

For other algebraic operations (and most other purposes), the polar coordinate representation (**Argand diagram**) is more convenient:

$$z \;= x + i\,y \;\equiv\; r(\cos\theta + i\,\sin\theta) \;\Rightarrow\; r \;=\; \sqrt{x^2+y^2} \;,\; \theta \;=\; \arctan\left(\frac{y}{x}\right) \quad . \tag{3}$$

Plot: Polar Coordinate/Argand Diagram Geometry

∗ Note that θ is multi-valued unless we restrict it to an inverval of length 2π (e.g. $[0, 2\pi]$).

∗ The polar coordinate $r = |z|$ is called the **magnitude** or **modulus** of z, and θ is referred to as the **argument** or **phase** of z.

- Defining the **complex conjugate** $z^* = x - iy$ of z, then the square of the magnitude of z is $z^*z \;=\; x^2 + y^2 \;=\; r^2$. The conjugate is useful in a multiplicative role in rendering complex denominators in real form.

- For further convenience, we state **Euler's formula**

$$\cos\theta + i\sin\theta \;=\; e^{i\,\theta} \quad , \tag{4}$$

which can serve as the definition of the exponential function (when the θ domain is extended to the entire complex plane). It can be proved using Taylor series expansions, once these have been defined for complex numbers.

Argand Diagram in Complex Plane

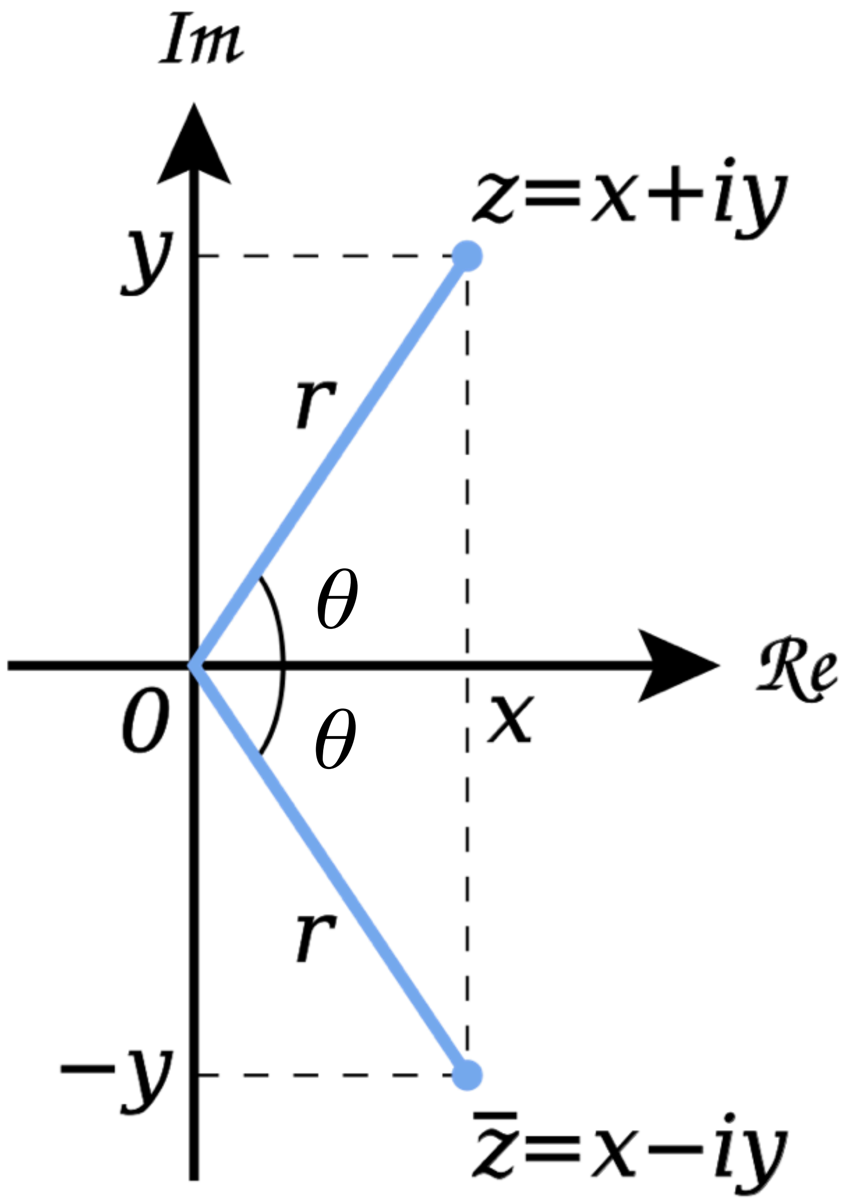


Diagram courtesy of Wikimedia Commons

• The polar coordinate form facilitates multiplication, division and inversion:

$$\begin{aligned} \text{Multiplication:} \quad z_1 * z_2 &= r_1 r_2 \, e^{i\,(\theta_1+\theta_2)} \\ \text{Division:} \quad z_1/z_2 &= (r_1/r_2)\, e^{i\,(\theta_1-\theta_2)} \quad [r_2 \;=\; |z_2| \;\neq\; 0] \\ \text{Raising to a power:} \quad z^\alpha &= r^\alpha \, e^{i\,\alpha\theta} \quad . \end{aligned} \tag{5}$$

here, α is not necessarily an integer, and can even be a complex number.

• If we raise the Euler formula to the n^{th} power, where n is an integer, we recover **de Moivre's formula**

$$\cos n\theta + i\,\sin n\theta \;=\; (\cos\theta + i\,\sin\theta)^n \quad , \tag{6}$$

which generates a host of well-known trigonometric identities in compact fashion (usually equating real and imaginary parts).

• **Example 1:** Consider A/W problem 6.1.8, summation of two trigonometric series: let

$$S_{\mathrm{R}} \;=\; \sum_{n=0}^{\infty} p^n\,\cos nx \quad , \quad S_{\mathrm{I}} \;=\; \sum_{n=0}^{\infty} p^n\,\sin nx \quad . \tag{7}$$

Then form

$$S \;=\; S_{\mathrm{R}} + i\,S_{\mathrm{I}} \;=\; \sum_{n=0}^{\infty} p^n\,e^{i\,nx} \;=\; \sum_{n=0}^{\infty} \left(p\,e^{i\,x}\right)^n \;=\; \frac{1}{1 - p\,e^{ix}} \quad , \tag{8}$$

a geometric series summation that is valid *iff* $|p\,e^{ix}| \equiv |p| < 1$. Since $p\,e^{-ix}$ is the conjugate of $p\,e^{ix}$, it follows that

$$S \;=\; \frac{1 - p\,e^{-ix}}{1 + p^2 - p\,(e^{ix} + e^{-ix})} \;=\; \frac{(1 - p\,\cos x) + i\,p\,\sin x}{1 - 2p\,\cos x + p^2} \quad . \tag{9}$$

Isolating the real and imaginary parts generates the required identities:

$$S_{\mathrm{R}} \;=\; \frac{1 - p\,\cos x}{1 - 2p\,\cos x + p^2} \quad , \quad S_{\mathrm{I}} \;=\; \frac{p\,\sin x}{1 - 2p\,\cos x + p^2} \quad . \tag{10}$$

Eq. (7) constitutes Fourier series expansions for these functions.

2.1 Complex Functions of a Complex Variable

Complex functions generally can be written in the form

$$w(z) \;=\; u(x,\, y) + i\, v(x,\, y) \quad , \tag{11}$$

implying two separate functions $(u,\, v)$ of the two real variables $(x,\, y)$. The problem of graphical representation is usually solved by "mapping" points and curves from the $z = x + i\,y$ to the $w = u + i\,v$ plane.

• Simple examples of mappings are *addition* (= translation), *multiplication* (= scaling + rotation), and *inversion* (= inversion + reflection). **A & W, Sec. 6.7**

Plot: Mappings of Addition, Multiplication and Inversion

∗ Observe that among these, only addition preserves the shapes of curves.

• A more involved example is provided by $w = z^2 = r^2 e^{2i\theta}$; it maps the upper half z plane onto the *entire* w plane (as does the lower half z plane).

∗ The semi-circle with center at the origin maps onto a full circle centered at the origin. However, straight lines $x = c$ map over to hyperbolae, and select hyperbolae map over to straight lines in the w plane:

Plot: Mappings for $w = z^2$

• The inverse mapping $w = \sqrt{z}$ therefore possesses a multi-valued problem, which is solved by defining **branches** of the square-root function:

$$w_1(z) \;=\; \sqrt{r}\, e^{i\,\theta/2} \qquad \text{and} \qquad w_2(z) \;=\; \sqrt{r}\, e^{i\,(\theta+2\pi)/2} \quad . \tag{12}$$

This splits the z plane into two **Riemann sheets**, one for each branch. The sheets are joined at a **branch line** (or *branch cut*) extending from $z = 0$ to $z = \infty$ in any direction (though often along the positive real axis).

• Similarly, $w(z) = z^{1/n}$ is n-fold multi-valued for integer n. In this case, one defines a **principal branch** $w(z) = r^{1/n} e^{i\,\theta/n}$ as a single-valued function.

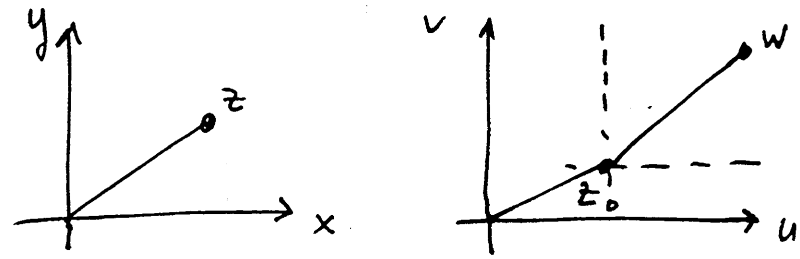
• Likewise, $\log_e(z) = \log_e r + i\,(\theta + 2n\pi)$ is infinitely multi-valued. Defining a principal branch $\log_e(z) = \log_e r + i\,\theta$ on $0 \le \theta \le 2\pi$, it is single-valued.

Trivial examples of mapping:

1. Addition ($\rightarrow$ translation)

$$w(z) = z + z_0$$

$$(z_0 = x_0 + iy_0)$$

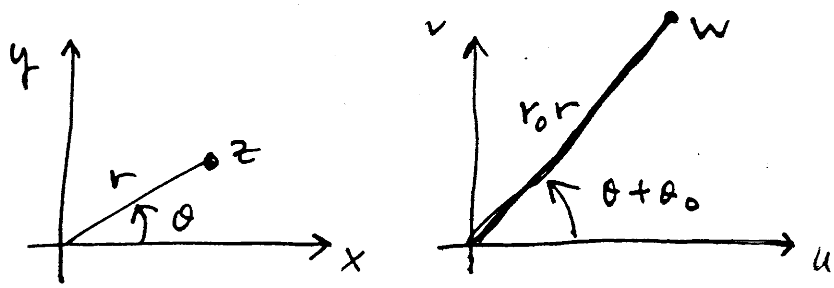


2. Multiplication ($\rightarrow$ scaling + rotation)

$$w = z z_0$$

$$(z_0 = r_0 e^{i\theta_0})$$

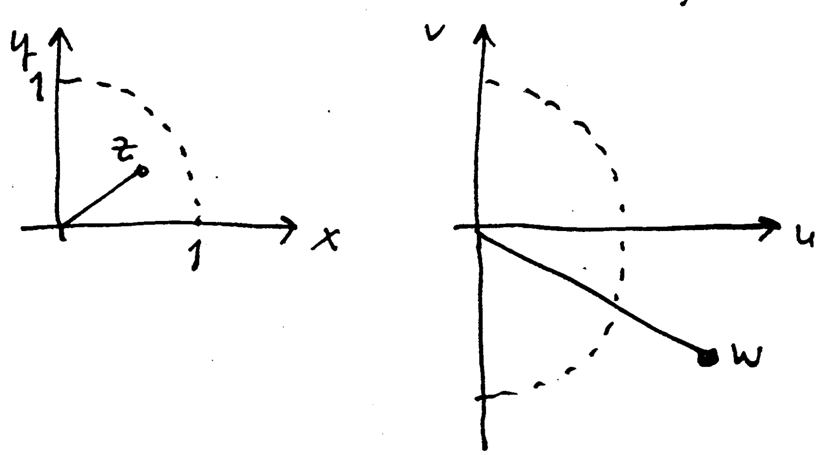
$$w = r_0 r e^{i(\theta + \theta_0)}$$



3. Inversion ($\rightarrow$ inversion + reflection)

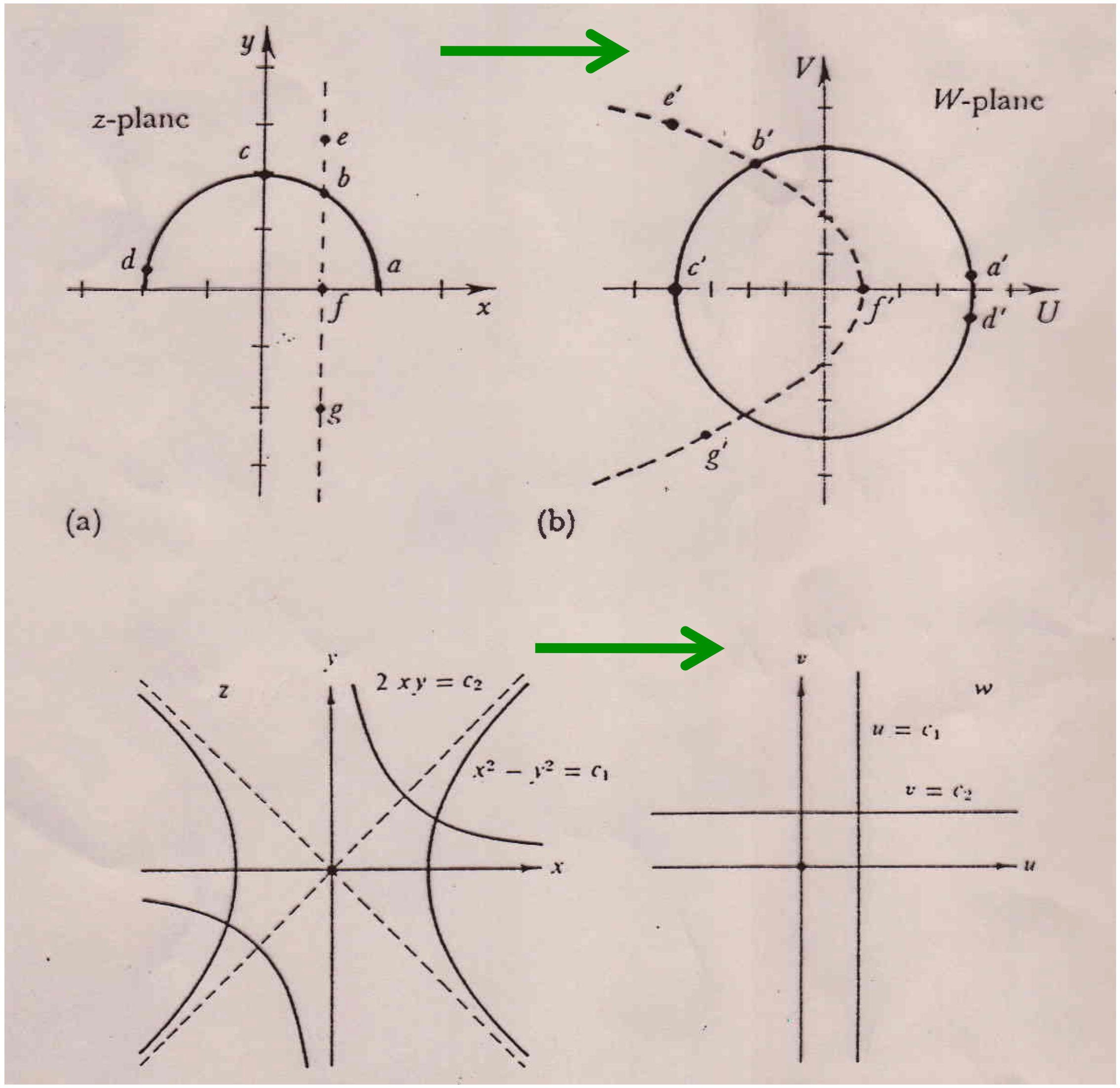
$$w = 1/z$$

$$= \frac{1}{r} e^{-i\theta}$$



Note even these trivial examples (except #1) do not preserve shape of curves.

Complex Plane Mapping $w=z^2$



From Mathews & Walker: *Mathematical Methods of Physics*