This applies when radiation dominates, and the perturbation evolution ODE becomes

$$\frac{d^2\delta}{dt^2} + \frac{1}{t}\,\frac{d\delta}{dt} - \frac{\Omega_{\rm rad}}{t^2}\,\delta \;=\; 0 \quad . \tag{13}$$

For power-law trial solutions this solves almost trivially as

$$\delta \;\propto\; t^{\kappa} \quad , \quad \kappa \;=\; \pm\sqrt{\Omega_{\rm rad}} \quad . \tag{14}$$

Contrasting this with the matter solution, we see that perturbations grow slowly during the radiation-dominated epoch (since $\sqrt{\Omega_{\rm rad}} \sim 10^{-2}$) and accelerate somewhat during the matter-dominated one. This "tango" does provide an imprint on the CMB as well as galaxy formation.

2 Jeans Instability

The focus thus far has been on the time development, yet has avoided any discussion of the spatial scale of the perturbations. The influence of pressure has also been ignored. Now we attend to both. The pressure contributions can be managed by introducing an **equation of state**, as we have done before. The simplest form of EOS that is expedient for a linear perturbation analysis is the adiabatic one, expressed using the **sound speed** $c_s(\rho)$:

Longair Sec. 11.4

$$c_s^2 \;=\; \frac{\partial P}{\partial \rho} \;\propto\; \rho^{\gamma-1} \quad , \tag{15}$$

for an adiabatic index γ. Since this is a function of density, it depends both on the scale factor (for non-relativistic matter with $\gamma = 5/3$ it yields $c_s^2 \propto \rho^{2/3} \propto (1+z)^2$), and on the local, peculiar perturbation scale. We introduce such scales via

$$\delta(\mathbf{x},\, t) \;=\; \sum_k \delta_k(t)\, \exp\bigl\{ i\mathbf{k}\cdot\mathbf{x}\bigr\} \quad , \tag{16}$$

a Fourier decomposition of the density fluctuations. Thus, $\mathbf{k}$ represents a perturbation wavenumber, defining a **power spectrum** $|\delta_k|^2$ of fluctuations. Clearly, $\lambda = 2\pi a/k$ (with $k = |\delta_k|$) represents the physical scale of the fluctuations for a scale factor $a = 1/(1+z) \leq 1$. Therefore, we expect in intricate interplay between growth rates, scales and redshift.

• The perturbation evolution equation was derived by taking a gradient of the Euler equation; this force equation contains a gradient of the pressure: $(1/\rho)\,\nabla(\delta P) - \nabla(\delta\Phi)$ is the acceleration. Thus, the introduction of finite pressure amounts to adding a term in Poisson's equation that is proportional to $(1/a^2)\,\nabla^2(\delta P)$, i.e., a second derivative, with $\delta P \approx c_s^2 \delta\rho$. It therefore becomes a simple matter using

$$\nabla^2 \delta(\mathbf{x},\, t) \;=\; -\sum_k \delta_k(t)\, |\mathbf{k}|^2 \,\exp\bigl\{ i\mathbf{k}\cdot\mathbf{x} \bigr\} \quad , \tag{17}$$

and $k = |\mathbf{k}|$, to write down

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\,\frac{\dot{a}}{a}\,\frac{\partial \delta_k}{\partial t} \;=\; \biggl\{ 4\pi G\rho_{\rm m} - \frac{c_s^2 k^2}{a^2} \biggr\}\, \delta_k \tag{18}$$

as the **scale-dependent fluctuation evolution equation** in Fourier space, *returning again to the matter-dominated case*. Given $\lambda = 2\pi a/k$, the long wavelength domain simply reproduces the global matter-driven perturbation evolution discussed in Section 1.1. A density-dependent characteristic wavelength naturally emerges from this ODE:

$$\lambda_{\rm J} \;\equiv\; \frac{2\pi a}{k_{\rm J}} \;=\; c_s \sqrt{\frac{\pi}{G\rho_{\rm m}}} \sim c_s t_{\rm ff} \quad , \tag{19}$$

where $t_{\rm ff}$ is the familiar free-fall collapse timescale for a self-gravitating system. This $\lambda_{\rm J}$ scale is referred to as the **Jeans length**, and the evolution equation then takes the form **C & O, Sec. 12.2**

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\,\frac{\dot{a}}{a}\,\frac{\partial \delta_k}{\partial t} \;=\; (2\pi c_s)^2 \biggl\{ \frac{1}{\lambda_{\rm J}^2} - \frac{1}{\lambda^2} \biggr\}\, \delta_k \quad , \tag{20}$$

with solutions varying according to the dominance of gravity (i.e., coming from the $1/\lambda_{\rm J}^2$ term as before) or pressure (the new $1/\lambda^2$ term).

• If $\lambda > \lambda_{\rm J}$, clearly gravity dominates pressure buoyancy, and sound waves cannot communicate over the size of the region to disrupt the pull of gravity: the collapse timescale $t_{\rm ff}$ is shorter than the time λ/c_s for sound waves to propagate. The solutions are as before. This domain is more likely realized for low sound speeds or really dense matter, or both.

• If $\lambda < \lambda_{\rm J}$, pressure buoyancy drives the system. The fluctuation equation is (at early times) dominated by two terms, and can be written

$$\frac{\partial^2 \delta_k}{\partial t^2} \approx -\frac{4\pi^2 c_s^2}{\lambda^2}\,\delta_k \quad \Rightarrow \quad \delta_k(t) \propto \sin\frac{2\pi c_s t}{\lambda} \quad . \tag{21}$$

The solution is thus oscillatory, i.e. a "standing" sound wave. Yet as the Universe ages, the linear derivative term becomes influential, and this introduces an exponentially decaying or **damping** modification to the oscillation.

∗ This pressure versus gravity scale dichotomy has an analog in considerations of star formation, and also late stage evolution of stars.

Plot: Temporal evolution of scale-dependent fluctuations (Peebles)

• One would like to know the redshift evolution of this dichotomy in the age domain $z_{\rm eq} \gtrsim z \gtrsim z_{l.s}$. Then, the matter-radiation interaction ties the matter temperature T to that of the radiation, and one has **isothermal perturbations**. Using the ideal gas equation of state, $c_s^2 \sim kT/m_p \propto (1+z)$. Also, the baryonic density evolution is given by $\rho_{\rm b} = \Omega_{\rm b}\rho_c(1+z)^3$. Accordingly, the Jeans length for purely baryonic matter scales as

$$\lambda_{\rm J} = \sqrt{\frac{\pi kT}{G\rho_{\rm b} m_p}} \approx \frac{7.7\,{\rm kpc}}{(1+z)\sqrt{\Omega_{\rm b}h^2}} \quad . \tag{22}$$

Even with perturbations seeded by dark matter, i.e. $\Omega_{\rm b} \to \Omega_m \approx 0.27$, this establishes that density fluctuations in the recombination era should be on scales of $\sim 10 - 30\,{\rm pc}$, *corresponding to very compact regions*!

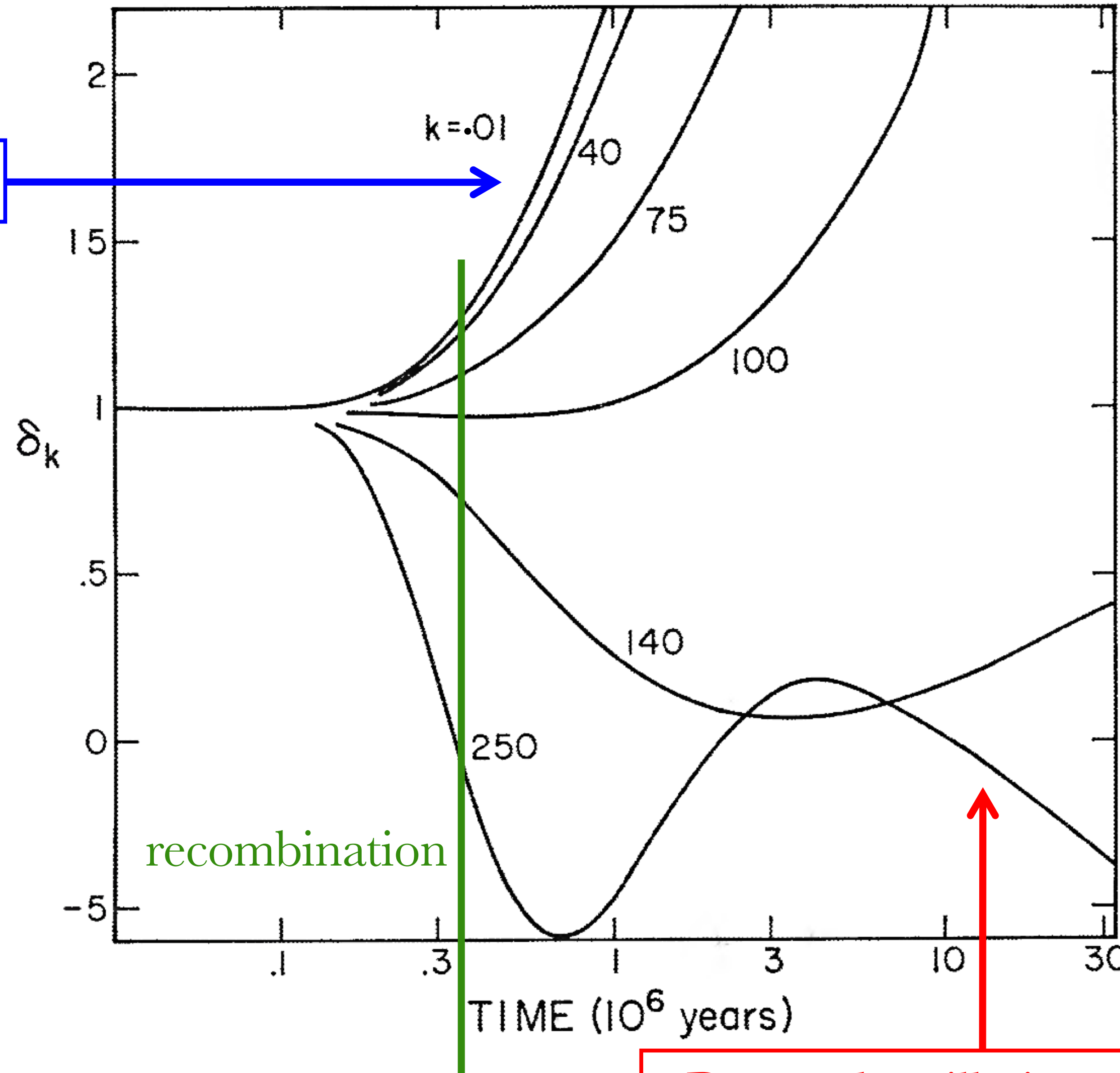
This can be expressed as an angular scale $\theta_{\rm J} = \lambda_{\rm J}/d_A$ using the angular diameter distance d_A in a matter-dominated Universe. Thus,

$$d_A \equiv \frac{c}{H_0(1+z)} \int_0^z \frac{dz'}{E(z')} \approx \frac{3.0\,{\rm Gpc}}{\sqrt{\Omega_m h^2}} \left\{ 1 - \frac{1}{\sqrt{1+z}} \right\} \tag{23}$$

since $E(z) \approx \sqrt{\Omega_m}(1+z)^{3/2}$. At recombination ($z \sim 1100$), the Jeans angular scale is then $\theta_{\rm J} \approx 2.6 \times 10^{-6}(1+z) \sim 0.16^\circ$. This is substantially smaller than the principal angular scale of $\sim 1^\circ$ for CMB fluctuations in the WMAP and Planck measurements. *This defines the key* **horizon problem**.

Isothermal Perturbation Growth

Power-law $t^{2/3}$ growth



Damped oscillation

- Baryon density fluctuation evolution for different scale wavenumbers k (in units of 10^{-20} cm^{-1}).
- Peebles (1969, ApJ **157**, 1075).

$k = 2\pi\, a/\lambda$

• One can ascribe a mass to the density fluctuation scale and this is termed the **Jeans mass**: **C & O, pp. 1250-1**

$$M_{\rm J} \;=\; \frac{4\pi}{3}\,\rho_{\rm m}\left(\frac{\lambda_{\rm J}}{2}\right)^3 \;\sim\; \frac{6.7\times 10^4 M_\odot}{\sqrt{\Omega_m h^2}} \;\sim\; 10^6 M_\odot \quad . \tag{24}$$

This mass scale is independent of redshift z, and it represents the mass scale above which gravity seeds density fluctuation growth in $z_{\rm eq} \gtrsim z \gtrsim z_{l.s}$ epochs. This defines the "onset" of structure formation: matter conglomerations that exceed $M_{\rm J}$ will continue to grow as gravity drives collapse.

$*$ The fact that the Jeans mass is comparable to globular cluster masses is suggestive that *cluster formation possibly pre-dates galaxy formation*.

After matter decouples from radiation, it cools fast as the Universe ages, and so $T \propto (1+z)^2$. This modifies the calculation and introduces redshift dependence to the Jeans mass. Then

$$\lambda_{\rm J} \;\propto\; \frac{1}{\sqrt{1+z}} \quad \text{and} \quad M_{\rm J} \;\sim\; \frac{6.7\times 10^4 M_\odot}{\sqrt{\Omega_m h^2}}\left(\frac{1+z}{1+z_{l.s}}\right)^{3/2} \quad . \tag{25}$$

Therefore, the Jeans mass is now redshift-dependent, and smaller than it was at recombination. This means that structure grown up to matter-radiation decoupling will continue to grow, not being disrupted by pressure buoyancy.

This perturbation analysis has acquired the time evolution of the fluctuations, but *it contains no information on their amplitude*, $|\delta_k|$; this must be obtained by detailed simulations of an evolving universe.

• Yet, observationally, we know that $\delta\rho/\rho \sim 1$ now. In the matter-dominated epoch, we have both $a(t) \propto t^{2/3}$ and $\delta \propto t^{2/3} \propto 1/(1+z)$. Thus

$$\delta_k(z = z_{l.s}) \;\sim\; 10^{-3} \quad , \tag{26}$$

and all **seed models** of structure formation prior to this, i.e. in the radiation-dominated epoch, *must generate this universal result*.

2.1 Radiation-Dominated Epoch

Well prior to recombination, the matter universe perturbation analysis discussed so far does not apply. Two aspects need modification, the inclusion of relativistic density and a relativistic equation of state. The first is achieved via the substitution $4\pi G\rho_{\rm b} \to 32\pi G\rho_{\rm rad}/3$, as before, and the second via the use of $c_s \to c/\sqrt{3}$. Thus, one arrives at the evolution equation

Longair Sec. 12.4.1

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\,\frac{\dot{a}}{a}\,\frac{\partial \delta_k}{\partial t} \;=\; \left\{ \frac{32\pi}{3} G\rho_{\rm rad} - \frac{k^2 c^2}{3a^2} \right\} \delta_k \tag{27}$$

for redshifts $z > z_{\rm eq}$. The corresponding Jeans length is

$$\lambda_{\rm J} \;=\; \frac{c}{\sqrt{3}} \sqrt{\frac{3\pi}{8G\rho_{\rm rad}}} \;=\; \frac{\pi c}{H_0 (1+z)^2 \sqrt{3\Omega_{\rm rad}}} \;\approx\; \frac{7.55\,{\rm Gpc}}{(1+z)^2 \sqrt{\Omega_{\rm rad}}} \quad . \tag{28}$$

Note that $\rho_{\rm rad} = U_{\rm rad}/c^2 = \Omega_{\rm rad}\rho_c (1+z)^4$ with $\Omega_{\rm rad} \sim 4.8 \times 10^{-5}$, so this yields roughly 1 Mpc for $z = 10^3 \sim z_{\rm rec}$. That this is *4-5 orders of magnitude larger than the matter result at recombination* reflects the vast difference in the EOS between radiation with $c_s = c/\sqrt{3}$ and matter with $c_s = \sqrt{kT/m_p} \sim 4 \times 10^{-5} c$. *The perturbations are no longer isothermal.*

∗ In the radiation-dominated epoch, the Jeans radius is a sizable fraction of the horizon scale. Thus, *sound waves propagate fast enough to largely smooth out density perturbations, so that one anticipates that the CMB will be mostly smooth on the sky.* This mostly solves the **horizon problem**.

• Concomitantly, one anticipates that the Jeans mass contained in baryons is also much larger:

$$M_{\rm J} \;=\; \frac{4\pi}{3}\,\rho_{\rm b} \left(\frac{\lambda_{\rm J}}{2} \right)^3 \;=\; \frac{\pi^3}{6\sqrt{3}}\,\frac{c^3}{GH_0}\,\frac{\Omega_{\rm b}}{\Omega_{\rm rad}^{3/2}} \;\sim\; \frac{2.6 \times 10^{23} M_{\odot}}{(1+z)^3}\,\frac{\Omega_{\rm b}}{\Omega_{\rm rad}^{3/2}} \quad . \tag{29}$$

At $z = 10^4 \sim z_{\rm eq}$ this is of the order of $3 \times 10^{16} M_{\odot}$, a mass that is around 30-50 times that of the Virgo and Coma clusters of galaxies.

Plot: Evolution of Jean's mass with redshift/scale factor

• The combination of radiation and matter contributions sets up the $\sim 1^\circ$ angular scale, and lowers $M_{\rm J}$ to the levels pertaining to large clusters.

Jeans Mass Evolution

- Evolution of Jeans mass M_J with scale factor in radiation and matter-dominated epochs.
  - Observe the abrupt drop in M_J at the z_{eq} boundary.
- Also shown are the horizon scale mass M_H and a mass M_S associated with electron-photon scattering scales (Silk damping).
- Fig. 12.1 of Longair book.
- Note the high value of z_{eq} in this older presentation.