helium is an important factor. This destruction of deuterium is what yields the sensitivity of n_d to $\Omega_{\rm b} h^2$ so that it becomes a useful diagnostic.

The peak abundance of deuterium is not as sensitive to $\Omega_{\rm b} h^2$ as Eq. (24) implies, largely because adjustments in this cosmological parameter are accompanied by adjustments to the temperature/redshift for neutron freeze-out.

$\boxed{\textbf{Plot:}}$ Schramm and Wagoner History of Light Element Abundances

- Observe that the $n_d = n_n$ *crossover* occurs at $T \sim 8 \times 10^8\,$K, i.e. at time $t \approx 3$ minutes. *This identifies the age of the Big Bang nucleosynthetic epoch.*

2.2 Helium and Lithium Production

Longair, Sec. 10.3

As the mass of the light elements increases, the complexity of the network of involved reactions becomes greater. For helium, deuterium is destroyed, in a manner similar to in the pp chain in stellar interiors:

$$\begin{aligned} d + d &\leftrightarrow t + p \\ d + d &\leftrightarrow {}^3He + n \\ t + d &\leftrightarrow {}^4He + n \\ {}^3He + d &\leftrightarrow {}^4He + p \end{aligned} \tag{25}$$

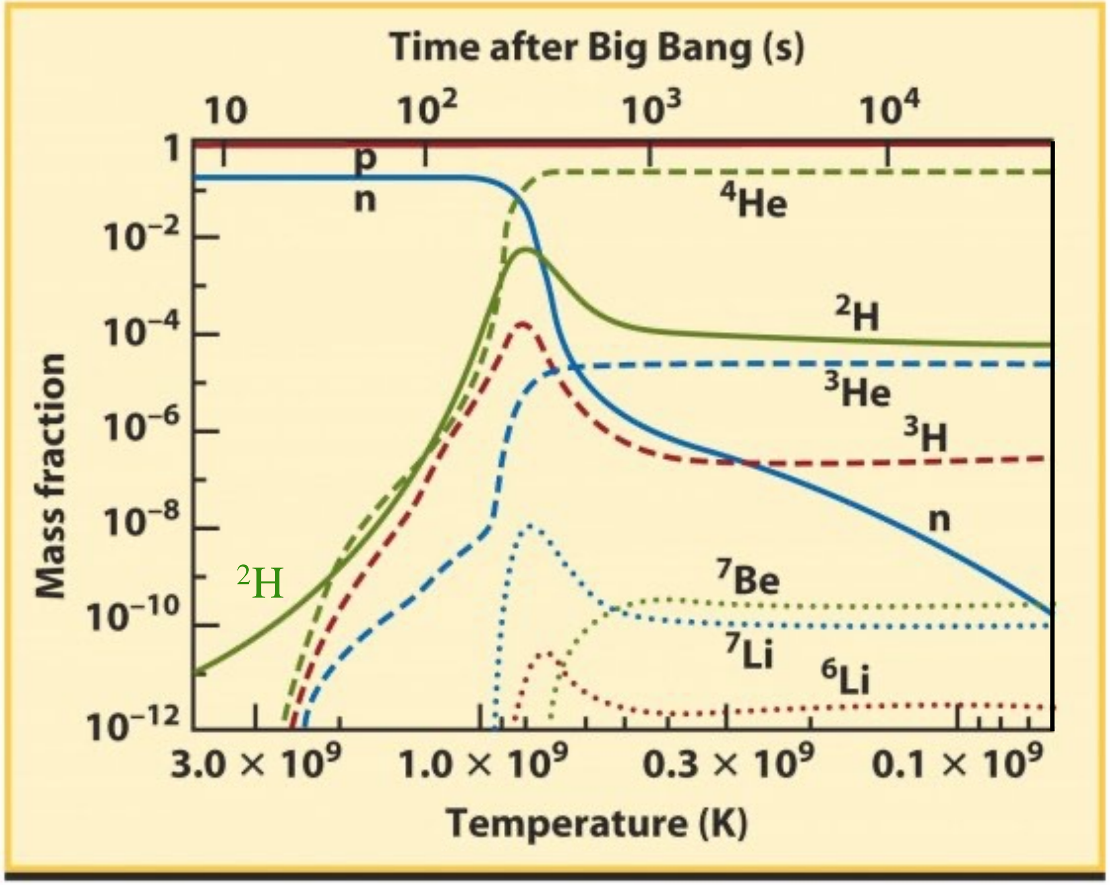
The stability of alpha particles (7 MeV/nucleon as opposed to 1.1 MeV/nucleon for deuterium) drives the "equilibrium" toward ${}^4He$ production.

$*$ The production of helium depletes the Universe of deuterium and tritium, so their abundances decline substantially until helium freeze-out occurs.

$\boxed{\textbf{Plot:}}$ Schramm and Wagoner History of Light Element Abundances

As the universe evolves, while the <u>eventual</u> d abundance is sensitive to $\Omega_{\rm b} h^2$, that for ${}^4He$ is not. The reason for this is essentially caused by the n_d/n_n ratio at the peak deuterium abundance being insensitive to $\Omega_{\rm b} h^2$, and the predominant delivery of helium is triggered near the epoch of peak deuterium.

Big Bang Light Element Synthesis



- Fractional abundances of light elements and their evolution through the nucleosynthetic era. Relative abundances depend on $\Omega_b h^2$ parameter so that measured final apportionment probes the baryonic matter content.
- Adapted from the original version in the seminal review of Schramm & Wagoner (1977; Ann. Rev. Nucl. Sci. 27, 37), Figure 3 therein.

- In the earlier Universe, ν decoupling at a temperature of $T_{10} \sim 3$ terminates "free" inverse beta decay reactions and establishes

$$\frac{n_n}{n_p} \sim 0.16 \approx e^{-Q_n/kT} \tag{26}$$

prior to the onset of rapid helium production. The eventual ${}^4He$ abundance dominates that of deuterium and tritium, so that it is **determined by neutron "conservation"** in either hydrogenic or helium form:

$$\begin{aligned} n_H &= n_p - n_n \\ n_{He} &= (n_p + n_n) - (n_p - n_n) = 2n_n \quad . \end{aligned} \tag{27}$$

This constraint can be imposed because the age of the Universe is insufficient for free neutron decay to take hold. This then yields a helium mass fraction:

$$Y = \frac{n_{He}}{n_p + n_n} = \frac{2n_n}{n_p + n_n} \approx 28\% \quad . \tag{28}$$

The observed value, from UV observations of the ISM, is $24\,\%$.

- Lithium production is sensitive to $\Omega_{\rm b} h^2$, largely because it has a higher binding energy (not per nucleon). Hence, relevant reactions are in the $e^{-E_B/kT}$ tail more than for deuterium and helium.

Plot: Light Element Abundance Constraints on Baryonic Matter

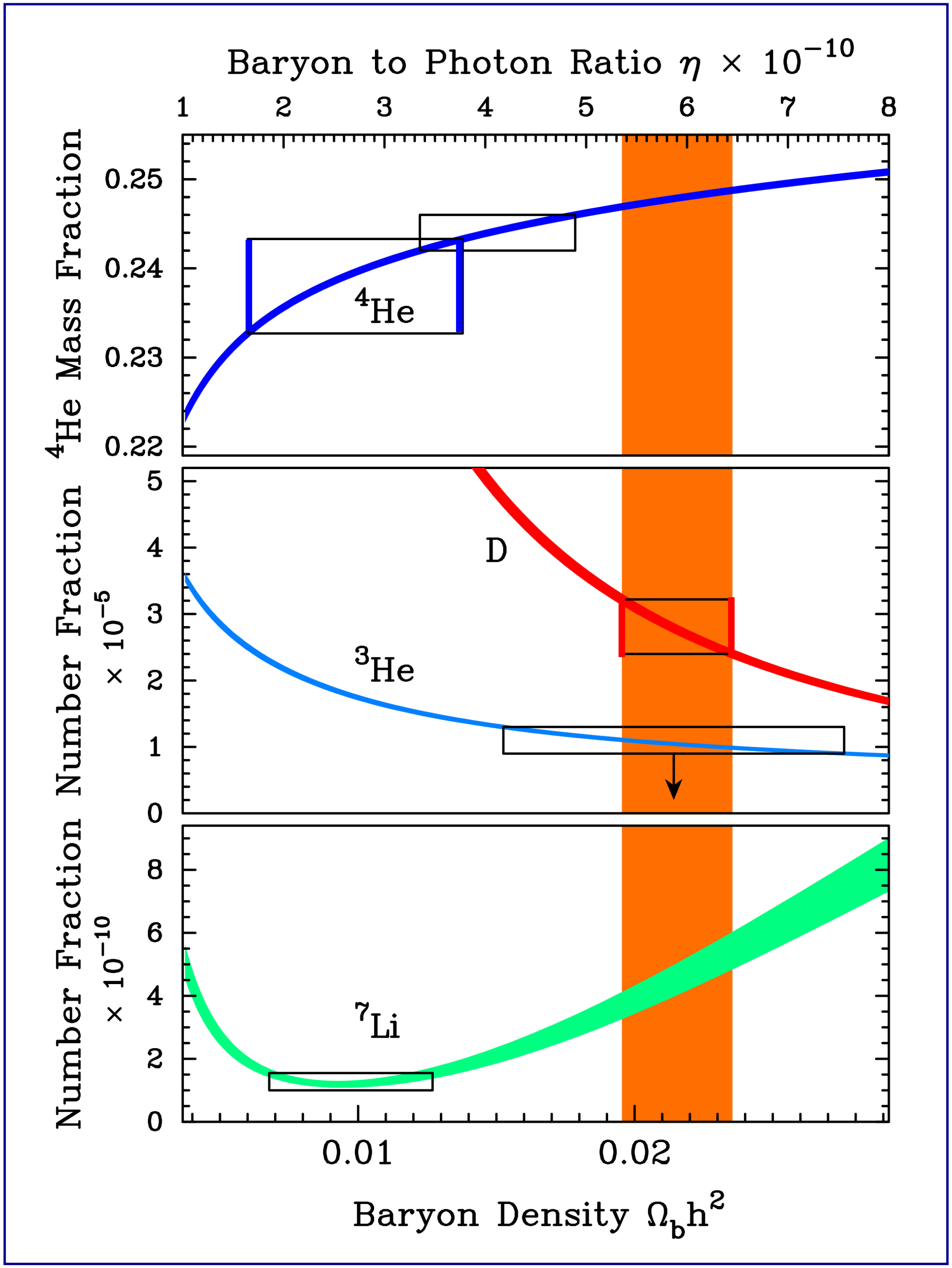
The bottom line is that these light element constraints concur and establish a tight bound on the density of baryonic matter in the universe:

$$\boxed{0.008 \lesssim \Omega_{\rm b} h^2 \lesssim 0.025 \quad ,} \tag{29}$$

which must be matched by subsequent determinations of the cosmological parameters of the universe. This was a test that WMAP was subjected to, and passed admirably, since it determined $\Omega_{\rm b} h^2 \sim 0.022$.

Primordial Nucleosynthesis Diagnostics

- Comparison of four light element abundances, observed (denoted by vertical sides of boxes) and predicted (colored curves), as functions of baryon density.
- From Kirkman et al. (2003: ApJS **149**, 1) [arXiv: 0302006] who reported D abundances (shaded region constraints) along line of site to quasar QSO 1243+3047.



10. STRUCTURE EVOLUTION

Matthew Baring – Lecture Notes for ASTR 360, Spring 2025

1 Dynamics of Linear Density Perturbations

Longair, Sec. 11.2

The Λ-CDM models of the Universe presume that structure formation is seeded by gravity alone, without the assistance of other forces. Consider density perturbations $\delta\rho$ to the local mean baryonic density ρ_0. There are three fluid equations that govern the evolution of local perturbations:

$$\begin{aligned} \text{Euler EOM (force):} \qquad \frac{\partial \mathbf{v}}{\partial t} &= -\frac{\nabla P}{\rho} - \nabla\Phi - \bigl(\mathbf{v}\cdot\nabla\bigr)\mathbf{v} \\ \text{mass/energy conservation:} \qquad \frac{\partial \rho}{\partial t} &= -\nabla\cdot\bigl(\rho\mathbf{v}\bigr) \\ \text{Poisson's equation:} \qquad \nabla^2\Phi &= 4\pi G\rho - \Lambda \quad . \end{aligned} \tag{1}$$

Here, $\Phi(\mathbf{r})$ is the gravitational potential, and the gradient operator ∇ applies to physical coordinates $\mathbf{r}$. The cosmological constant Λ is included.

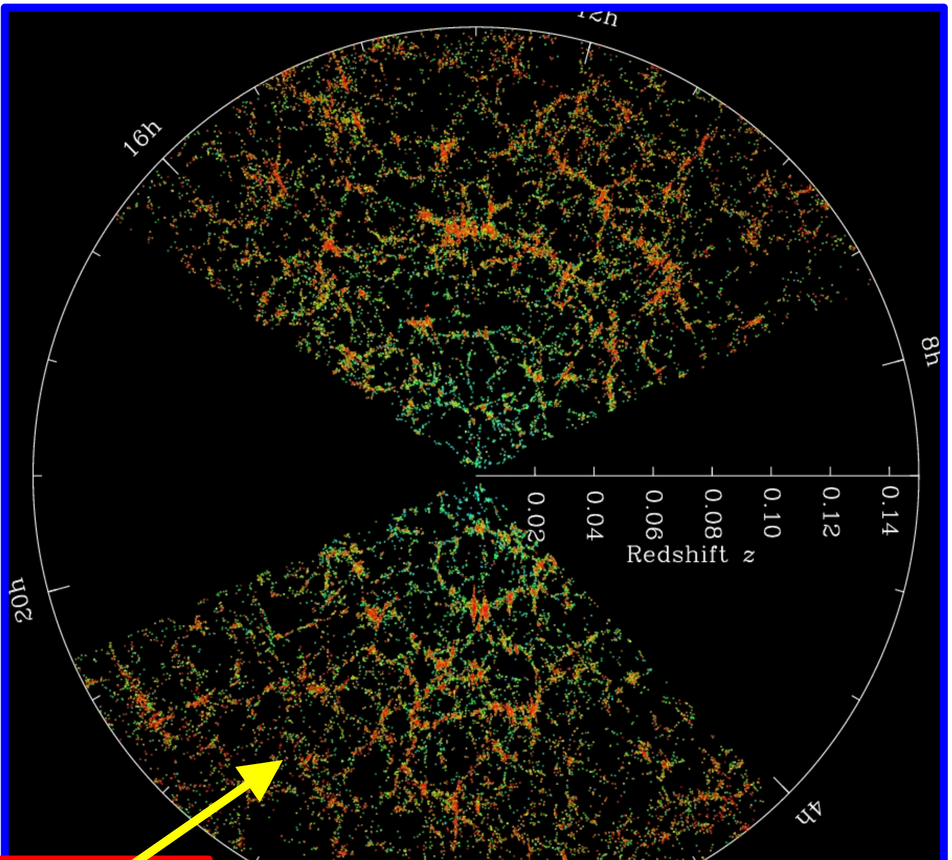
Plot: The Progression of Structure Formation

- Linearizing these equations about the means ρ_0 and Φ_0 and the velocity (Hubble flow) at any epoch can be achieved with the correspondences

$$\rho \;\to\; \rho_0 + \delta\rho \quad , \quad \mathbf{v} \;\to\; \mathbf{v}_0 + \delta\mathbf{v} \quad , \quad \Phi \;\to\; \Phi_0 + \delta\phi \tag{2}$$

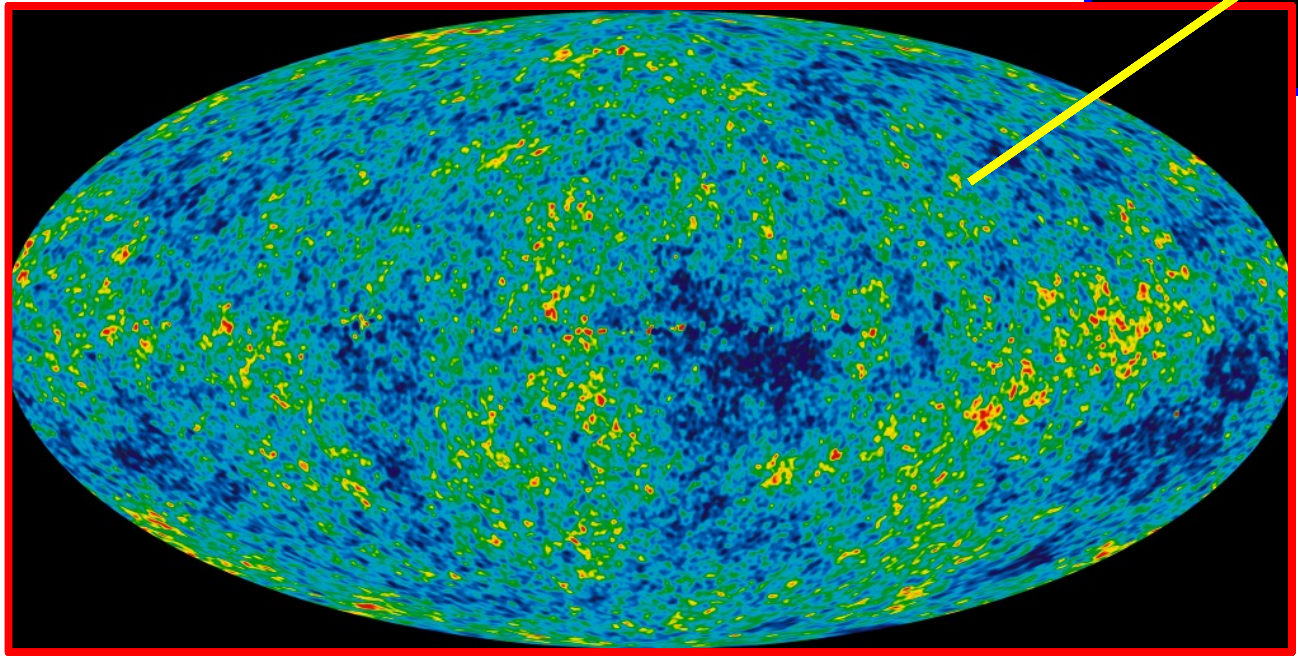
The resulting equations will apply after the Universe becomes matter dominated, i.e. at $z \lesssim z_{\rm eq} \sim 10^4$. Prior to this, radiation pressure is a homogenizing force, ironing out substantial perturbations (i.e. at the CMB level). Thus, structure formation mostly post-dates the recombination era.

The Progression of Structure



The SDSS view

The WMAP view



Models driven by gravity have to deliver these in sequence.

We will not solve this coupled system, as the derivation is long and involved. Instead, we will quote the resultant linearized solution. This is constructed in terms of a scaling of the perturbed density $\delta\rho \to \rho_{\rm m}(t)\,\delta(\mathbf{x},\, t)$, so that

$$\rho(\mathbf{r},\, t) \;=\; \rho_{\rm m}(t)\,\bigl[1+\delta(\mathbf{x},\, t)\bigr] \quad , \quad \rho_{\rm m} = \frac{\Omega_{\rm m}\rho_c}{\bigl[a(t)\bigr]^3} \quad . \tag{3}$$

With these transformations, one can recast the constituent equations for density/pressure perturbation evolution in relatively simple form. The leading order contribution is, of course, the average matter evolution, which generates the Friedmann equation. The $\delta \equiv \delta\rho/\rho_{\rm m}$ correction to this is described by a single <u>cold matter</u> **perturbation evolution equation**:

$$\boxed{\frac{\partial^2\delta}{\partial t^2} + 2\,\frac{\dot{a}}{a}\,\frac{\partial\delta}{\partial t} \;=\; 4\pi G\rho_{\rm m}\,\delta \quad .} \tag{4}$$

Therefore, the solutions to this second-order ODE depend on the cosmology adopted, which controls the dependence $a(t)$.

- The general solution for perturbation growth (and decay) was obtained by Carroll et al. (1992, ARAA), Eq. (28) therein, leveraging the general expression for $H(t) = \dot{a}/a$. Using the definition of $\rho_{\rm m}$ in Eq. (3), the evolution equation in Eq. (4) can be rewritten as

$$\frac{\partial^2\delta}{\partial t^2} + 2\,\frac{\dot{a}}{a}\,\frac{\partial\delta}{\partial t} \;=\; \frac{3\Omega_{\rm m}H_0^2}{2a^3}\,\delta \quad , \tag{5}$$

for matter-dominated evolution, and this yields a <u>growth</u> solution

$$\delta \;\equiv\; \frac{\delta\rho}{\rho_{\rm m}} \;=\; \frac{5\Omega_{\rm m}H_0^2}{2}\,\frac{\dot{a}}{a}\int_0^a \frac{da'}{(\dot{a}')^3} \quad . \tag{6}$$

This result can be verified via direct insertion into Eq. (5) on conjunction with Friedman's equation to clean up the time derivatives of a. Alternatively, since $\dot{a}/a = H_0\,E(z)$, one can write

$$\delta \;=\; \frac{5\Omega_{\rm m}}{2}\,E(z)\int_z^\infty \frac{(1+z')\,dz'}{\bigl[E(z')\bigr]^3} \quad . \tag{7}$$

Either form can be used to deduce the scaling $\delta \propto t^{2/3}$ in the matter-dominated case.

1.1 Matter and Radiation-Dominated Epochs

For purposes of illustration, consider a $\Lambda = 0$, matter-dominated Universe that is marginally bound, i.e. $\Omega_{\rm m} = 1$ now. This is an Einstein-de Sitter universe with $a(t) = [3H_0 t/2]^{2/3} = 1/(1+z)$, so that $\dot{a}/a = 2/(3t)$ and

Longair, Sec. 11.4.1

$$4\pi G\rho_{\rm m} \; = \; \frac{4\pi G\Omega_m \rho_c}{a^3} \; = \; \frac{2\Omega_{\rm m}}{3t^2} \quad . \tag{8}$$

The ODE for the amplitude $\delta = \delta\rho/\rho_{\rm m}$ is

$$\frac{d^2\delta}{dt^2} + \frac{4}{3t}\,\frac{d\delta}{dt} - \frac{2\Omega_{\rm m}}{3t^2}\,\delta \; = \; 0 \quad . \tag{9}$$

For $\Omega_{\rm m} = 1$ (critical) this clearly has two power-law-like solutions, namely

$$\delta(t) \; = \; \alpha t^{2/3} + \beta t^{-1} \quad . \tag{10}$$

The β term represents damping and so is not of particular interest. The α term is a growing perturbation that traces the scale of the Universe.

C & O, pp. 1249-50

If $\Omega_{\rm m} < 1$, then $\delta \propto t^{\kappa}$ trial solutions to Eq. (9) yield an algebraic equation for κ, with one growing solution and one decaying one:

$$\kappa(\kappa - 1) + 4\kappa/3 - 2\Omega_{\rm m}/3 \; = \; 0 \quad \Rightarrow \quad \kappa \; = \; \frac{\pm\sqrt{1 + 24\Omega_{\rm m}} - 1}{6} \quad . \tag{11}$$

Naively interpreted, for $\Omega_{\rm m} \sim 0.27$, this gives $\kappa \approx 0.29 \ll 2/3$, so that structure grows considerably slower than the Universe expands. This situation worsens if dark matter is neglected. Yet, with dark matter present, this linearized domain of seed growth is sufficient for **later non-linear stages** to develop to present the structures we see in the Universe today.

- The solution for more general FRW models is more difficult to obtain. For radiation-dominated universes prior to $z_{\rm eq}$, we have $a(t) \propto t^{1/2}$ so that $\dot{a}/a = 1/(2t)$. The gravity term can be modified via a substitution

Longair Sec. 11.6

$$\rho_{\rm m} \; \to \; \rho_m + \frac{U_{\rm rad}}{c^2} \quad \Rightarrow \quad 4\pi G\rho_{\rm m} \; \to \; \frac{32\pi}{3}\,G\rho_{\rm rad} \; = \; \frac{\Omega_{\rm rad}}{t^2} \quad . \tag{12}$$

This applies when radiation dominates, and the perturbation evolution ODE becomes

$$\frac{d^2\delta}{dt^2} + \frac{1}{t}\,\frac{d\delta}{dt} - \frac{\Omega_{\rm rad}}{t^2}\,\delta \;=\; 0 \quad . \tag{13}$$

For power-law trial solutions this solves almost trivially as

$$\delta \;\propto\; t^{\kappa} \quad , \quad \kappa \;=\; \pm\sqrt{\Omega_{\rm rad}} \quad . \tag{14}$$

Contrasting this with the matter solution, we see that perturbations grow slowly during the radiation-dominated epoch (since $\sqrt{\Omega_{\rm rad}} \sim 10^{-2}$) and accelerate somewhat during the matter-dominated one. This “tango” does provide an imprint on the CMB as well as galaxy formation.