1.2 Thermonuclear Reaction Rates

We turn our attention to the probabilities of nuclear reactions. Since nuclei are positively-charged, there is a strong Coulomb repulsive force that inhibits reactions. This must be overcome before an even stronger nuclear attractive force dominates on short length scales, namely $r \sim 10^{-13}$ cm.

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Plot: Nuclear Potential Energy Profile

• The determination of rates must account for *quantum tunnelling* through the Coulomb barrier. Heisenberg's uncertainty principle is $\Delta x \Delta p_x \gtrsim \hbar/2$, so that penetration on scales Δx can be achieved by particles of momenta $\hbar/\Delta x$. Nucleons (protons, deuterons, tritium, helium etc.) are not confined to points in space. This coupling is manifested in the de Broglie wavelength $\lambda = h/p$, which can be ported into an energy virialization equation:

$$\frac{p^2}{2\mu} \;=\; \frac{h^2}{2\mu\lambda^2} \;\sim\; \frac{Z_1 Z_2 e^2}{\lambda} \quad . \tag{6}$$

Herein, μ is the reduced mass of an interacting system, i.e. $m_p/2$ for a pp collision. This can be solved for λ to yield $1/\lambda \sim \mu Z_1 Z_2 e^2/(2\pi^2\hbar^2)$, and thereby establish an effective binding energy (**nuclear barrier**) scale

$$E_b \;=\; \frac{\mu Z_1^2 Z_2^2 e^4}{2\pi^2\,\hbar^2} \;\approx\; \frac{Z_1^2 Z_2^2}{\pi^2}\,\frac{\mu}{m_e}\,\chi_H \quad , \quad \chi_H \;=\; 13.6\,\text{eV} \quad . \tag{7}$$

By equating $E_b \sim kT$, one establishes a natural temperature scale for nucleosynthesis of $T \sim 10^7$ K. This value can easily deliver slow nuclear burning in stellar interiors; in the fledgling Universe, it needs to be much higher.

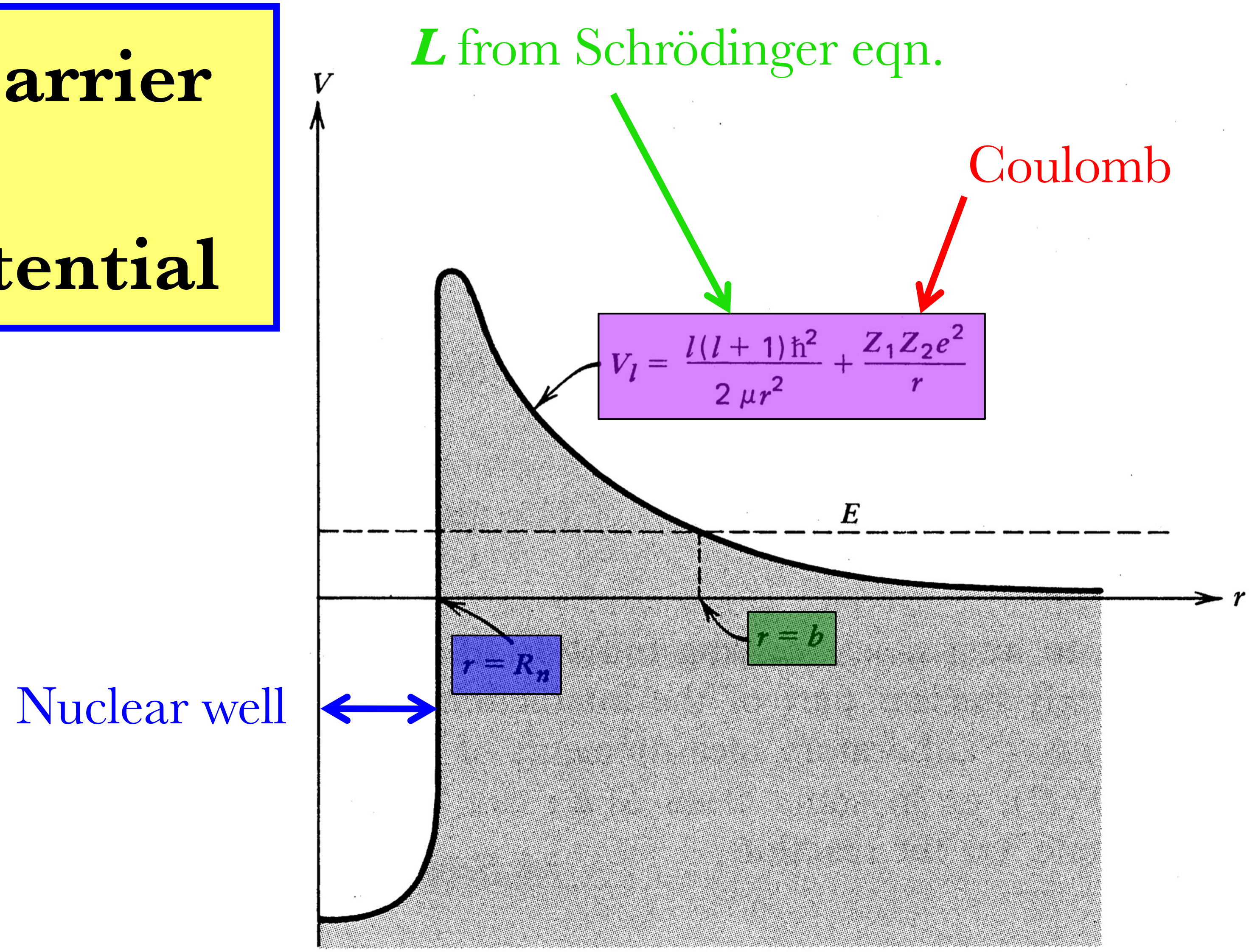
• To determine the behavior of the rates, we note that the cross section $\sigma(v)$ must be integrated over all v, with the probability of interaction being proportional to v. If n_v is the MB distribution of target nuclei, and n is the number density of "projectile" nuclei, then the rate of reaction is

$$\frac{dn}{dt} \;=\; \int_0^\infty n\, n_v\, \sigma(v)\, v\, dv \quad . \tag{8}$$

The most influential factor in this integral is the cross section. Crudely taking it to be the effective target area of interaction, we might expect

$$\sigma(v) \;\sim\; \pi\lambda^2 \;\propto\; \left(\frac{h}{p}\right)^2 \;\propto\; \frac{1}{E} \tag{9}$$

Coulomb Barrier and Nuclear Potential



- The effective potential V(r) governing the radial motion of one nucleus relative to another (Fig. 3.4 of Shapiro & Teukolsky). The short range attractive nuclear force dominates at $r<R_n$. At larger distances, the potential is dominated by the repulsive Coulomb force.
- The classical turning point for an orbit is at $r = b = Z_1Z_2e^2/E$ for a CM energy E.

in the nonrelativistic limit. However this is too small ($\sim 10^{-26}\,\mathrm{cm}^2$) and ignores tunnelling. Quantum tunnelling turns out to depend exponentially on the Coulomb barrier height U_c and the projectile energy $E = \mu v^2/2$:

$$\sigma(v) \;\propto\; e^{-2\pi^2 U_c/E} \;\propto\; \exp\left\{-\left(\frac{E_c}{E}\right)^{1/2}\right\} \quad , \quad E_c \;=\; \frac{2\pi^2\mu Z_1^2 Z_2^2 e^4}{\hbar^2} \tag{10}$$

for a Coulomb potential $U_c \sim Z_1 Z_2 e^2/\lambda$ with $\lambda = h/p$, i.e., $U_c \propto \sqrt{E}$. Thus, $E_c = (2\pi^2)^2 E_b$, and it follows that the reaction rate assumes the form

$$\frac{dn}{dt} \;\propto\; \int_0^\infty s(v)\,\exp\left\{-\left(\frac{E_c}{E}\right)^{1/2} - \frac{E}{kT}\right\} dv \tag{11}$$

for thermal nuclei. Here $s(v) = v^3\sigma(v)\,\exp\{(E_c/E)^{1/2}\}$ is a slowly-varying function of speed v.

Plot: The Nuclear Interaction Gamow Peak

The convolution of the strong rise of the cross section with energy and the exponential decline of the MB distribution yields a strongly-peaked integrand (denoted the **Gamow peak**). This occurs at $E_{\mathrm{G}} \sim E_c^{1/3}\,(kT)^{2/3}$.

• The total rate depends sensitively on temperature. Integration by the method of **steepest descents** (this is left as an exercise) would then yield a rate proportional to

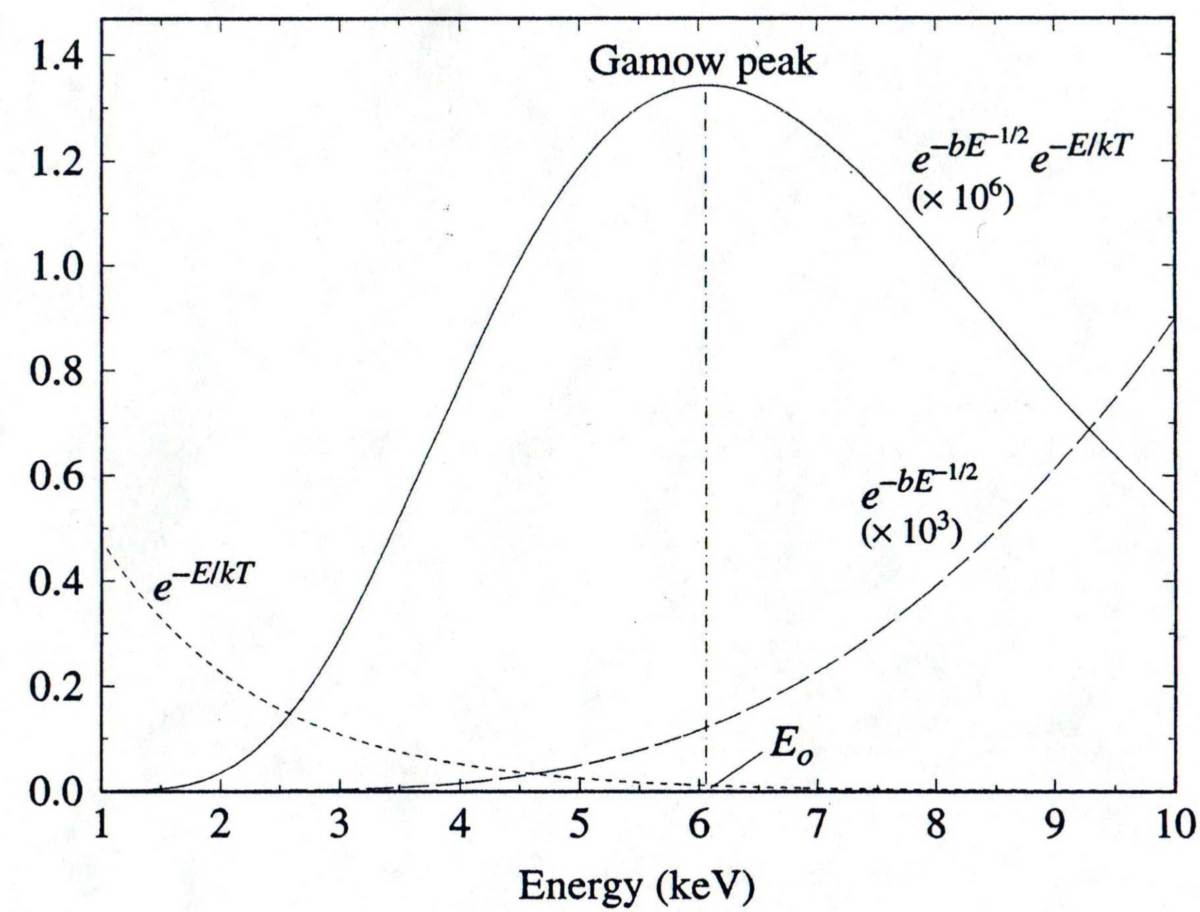
$$\exp\left\{-\left(\frac{E_c}{kT}\right)^{1/3}\right\} \tag{12}$$

Clearly, E_c depends on the composition of the gas. Yet, we can quickly write it down for hydrogen: $E_c = 2\pi^2\alpha_{\mathrm{f}}^2 m_p c^2 \approx 1\,\mathrm{MeV}$, where $\alpha_{\mathrm{f}} = e^2/\hbar c$.

$*$ For the Universe at a redshift of $z \sim 3 \times 10^8$, with a temperature of $T \sim 10^9\,\mathrm{K}$, the Gamow peak is in the 50–100 keV range, and $kT \ll E_c$ underpins the temperature sensitivity of the rate.

$*$ Accordingly, the nuclear interactions sample the exponential tails of Maxwell-Boltzmann distributions n_v of nuclear speeds.

Nuclear Interaction Gamow Peak



From: Carroll & Ostlie

- The Gamow peak arises from the competition between the Maxwell-Boltzmann exponential $e^{-E/kT}$ and the Coulomb barrier penetration factor $\exp(-bE^{1/2})$ in thermonuclear interactions.

1.3 Neutron Freeze-out

After neutrinos freeze out as the Universe ages, neutrons are the next species to decouple from the soup of nuclear interactions. The n/p ratio is controlled by thermodynamic equilibrium via a Boltzmann factor:

$$\frac{n_n}{n_p} \;=\; e^{-Q_n/kT} \quad , \quad Q_n \;=\; (m_n - m_p)c^2 \;=\; 1.2934 \;\; \text{MeV} \quad . \tag{13}$$

The balance energetically favors proton generation due to their lower energy/mass (i.e. chemical potential). This is why free neutrons decay.

∗ Hence **neutron freeze-out**, or decoupling, from the proto-nuclear soup occurs when kT drops below 1 MeV, i.e. for $T \lesssim 10^{10}\,$K.

Prior to the creation of light elements, the neutron abundance is described by a differential equation that expresses a non-equilibrium Saha-type calculation. The seminal work on this topic is the paper by Schramm & Wagoner (1977, Ann. Rev. Nucl. Sci. **27**, 37), hereafter SW. The neutron number density evolution satisfies

$$\frac{dn_n}{dt} \;=\; \lambda_p n_p - \lambda_n n_n \quad , \tag{14}$$

where the λ coefficients are given by

$$\lambda_n \;=\; \Lambda_{ne^+} + \Lambda_{n\nu} + \frac{1}{\tau_n} + \frac{1}{\tau_{exp}} \tag{15}$$

for the neutron absorption terms, free decay (e-lifetime of τ_n) and expansion losses (through τ_{exp}) due to density dilution, and

$$\lambda_p \;=\; \Lambda_{pe^-} + \Lambda_{p\bar{\nu}} \tag{16}$$

for the proton attrition (=neutron production) terms. In what follows, the expansion dilution term in Eq. (15) will be neglected. However, note that it is generally significant $\Rightarrow$ a dynamic, non-equilibrium evolution is appropriate.

∗ The production of deuterium via $n + p \leftrightarrow d + \gamma$ adds another term to both Eqs. (15) and (16); this is not detailed here, but discussed shortly.

• Schramm and Wagoner gave asymptotic forms for multitudinous nuclear reactions in the exponential (**quantum tunneling**) regime appropriate to relatively "low" temperatures. For example, with

$$\chi \;=\; \frac{m_e c^2}{kT} \;=\; \frac{5.93}{T_9} \quad , \tag{17}$$

with T_9 being the universe's temperature in units of $10^9\,$K, the interaction rate coefficients can be written

$$\begin{aligned} \lambda_n &\approx \frac{0.98}{\tau_n}\left\{1 + \frac{0.565}{\chi} - \frac{6.382}{\chi^2} + \frac{11.108}{\chi^3} + \frac{36.492}{\chi^4} + \frac{27.512}{\chi^5}\right\}\ \mathrm{s}^{-1} \\ \lambda_p &\approx \frac{0.98}{\tau_n}\left\{\frac{5.252}{\chi} - \frac{16.229}{\chi^2} + \frac{18.059}{\chi^3} + \frac{34.181}{\chi^4} + \frac{27.617}{\chi^5}\right\} e^{-2.531\chi}\ \mathrm{s}^{-1} \end{aligned} \tag{18}$$

These both display $\approx 1/\tau_n$ character that reflects neutron decay, though λ_p contains no neutron decay and so possesses an exponential factor. Note that the argument of this exponential is just $-Q_n/(kT)$, i.e. it couples to the binding energy of the neutron in Eq. (13).

Plot: The Nucleosynthetic Era: Wagoner, Fowler & Hoyle (1967)

- Clearly, as $\chi \to \infty$, the decay rate coefficients simplify, and we can write

$$\frac{\lambda_p}{\lambda_n} \;\approx\; \frac{5.25}{\chi}\, e^{-2.531\chi} \quad . \tag{19}$$

This is the equilibrium solution for a non-evolving n/p density ratio, from Eq. (14). It follows that neutron decline or freeze-out occurs when

$$\frac{n_n}{n_p} \lesssim 0.5 \quad \Leftrightarrow \quad \frac{\lambda_p}{\lambda_n} \lesssim 0.5 \quad \Leftrightarrow \quad T_9 \lesssim 6 \quad . \tag{20}$$

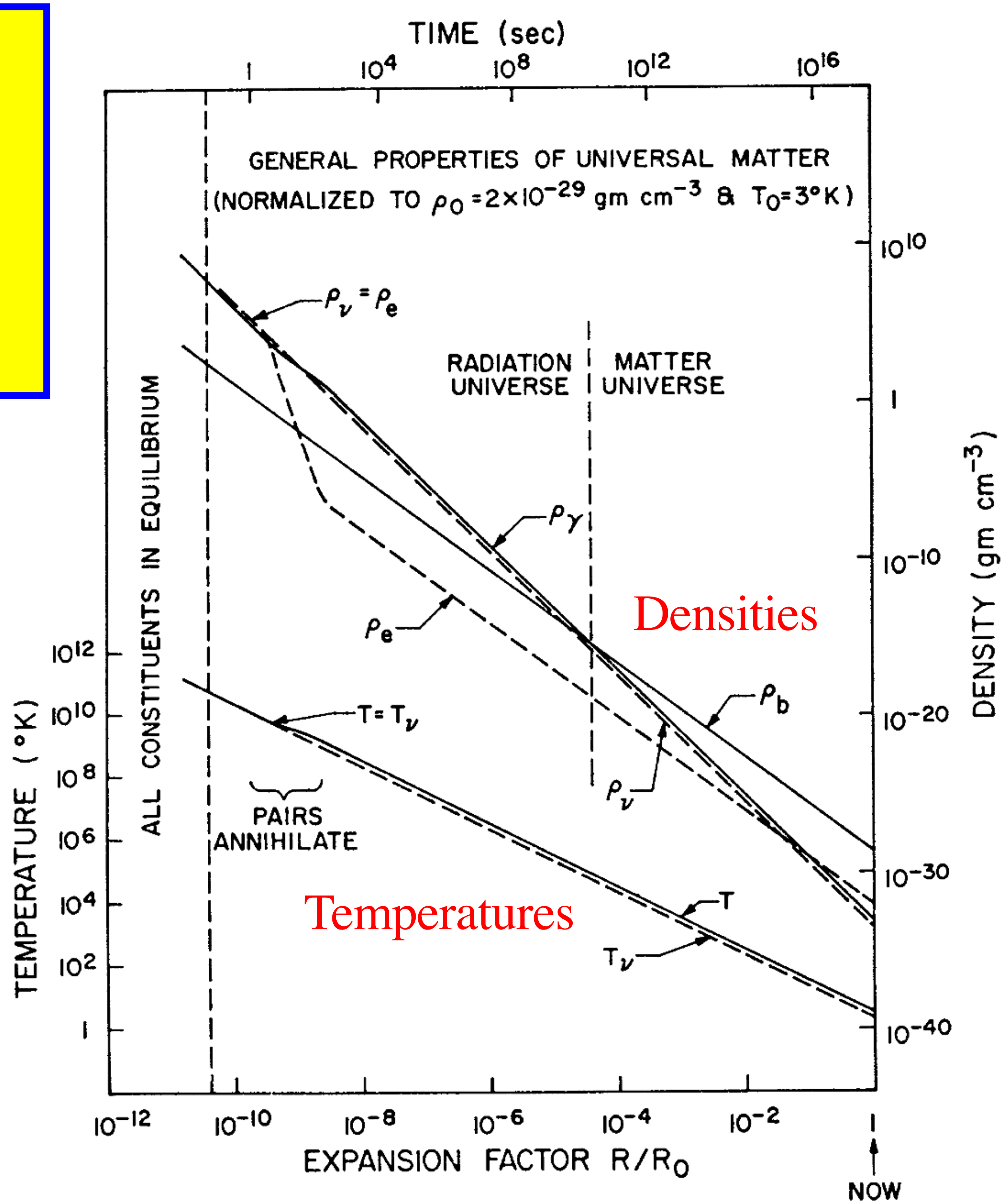
A more accurate dynamic calculation with full rate coefficients gives a lower value, $T_9 \sim 0.9$, corresponding to $t \sim 200\,\mathrm{sec} < \tau_n$.

Plot: Schramm and Wagoner History of Light Element Abundances

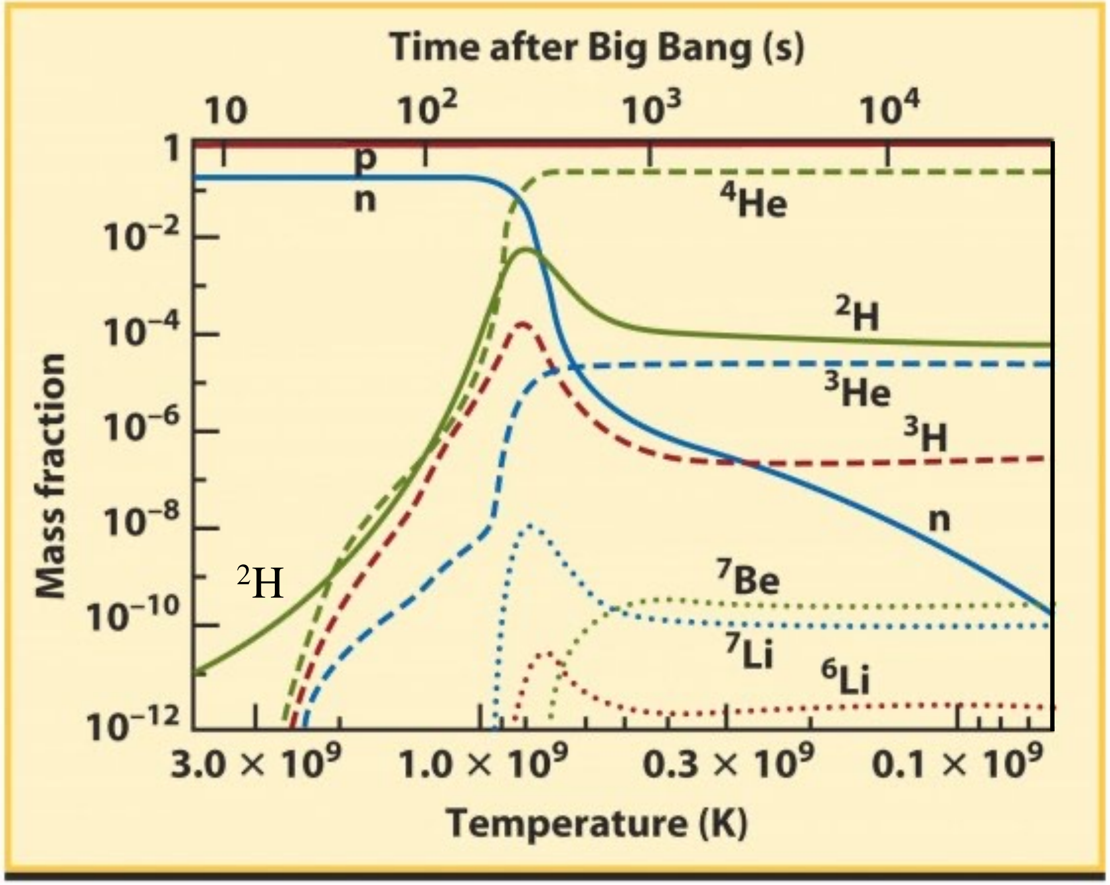
- Sub-exponential fall-off of n_n with replenishment from heavier elements (principally deuterium) precipitates an **interaction-driven decline**.

The Nucleosynthetic Era

- Densities and temperatures of baryons, electrons, neutrinos and photons in the nucleosynthetic epoch.
- From Wagoner, Fowler & Hoyle (1967, ApJ **148**, 3).



Big Bang Light Element Synthesis



- Fractional abundances of light elements and their evolution through the nucleosynthetic era. Relative abundances depend on $\Omega_b h^2$ parameter so that measured final apportionment probes the baryonic matter content.
- Adapted from the original version in the seminal review of Schramm & Wagoner (1977; Ann. Rev. Nucl. Sci. 27, 37), Figure 3 therein.

2 Light Element Synthesis

The coupling of the exponential dependence of reaction rates on the temperature and the binding energy of different nuclei yields a sensitivity that can be exploited to constrain the conditions of the Universe in the BBN epoch using present day light element abundances.

2.1 Deuterium Synthesis

Deuterium forms primarily via

$$n + p \;\leftrightarrow\; d + \gamma \quad , \tag{21}$$

and so its abundance is generated in the epoch of neutron freeze-out. The Saha equation-type (i.e., mimicking atomic ionization balance) thermal equilibrium value for the abundance of deuterium via this reaction is given by

$$\frac{n_p n_n}{n_d n_b} \;=\; \frac{4}{3n_b}\left(\frac{m_p m_n}{m_d}\right)^{3/2}\left(\frac{kT}{2\pi\hbar^2}\right)^{3/2}\exp\left\{-\frac{E_{B,d}}{kT}\right\} \quad , \tag{22}$$

where $n_b = n_p + n_n + n_d$ is the total baryonic number density, and

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$$E_{B,d} \;\equiv\; \big(m_n + m_p - m_d\big)c^2 \;=\; 2.23\,\text{MeV} \tag{23}$$

is the deuterium binding energy. Observe that this is less than the average binding energy per nucleon for helium.

$*$ Observe also that $E_{B,d} > Q_n$, thereby providing an added reason why neutron freeze-out terminates deuterium production: energetics favors it, so when the neutron supply fades, the deuterium production halts.

- The Saha equilibrium then becomes, for current baryonic number density $n_b = 1.12 \times 10^{-5}(\Omega_{\rm b}h^2)\,\text{cm}^{-3}$ with $h = H_0/[100\,\text{km/sec/Mpc}]$,

$$\frac{n_p n_n}{n_d n_b} \;=\; \exp\left\{25.82 - \log_e\left(\Omega_{\rm b}h^2 T_{10}^{3/2}\right) - \frac{2.58}{T_{10}}\right\} \quad . \tag{24}$$

Observe that this highlights $n_d \propto \Omega_{\rm b}h^2$ character, which differs from the Rowan-Robinson plot (to come) where absorption of deuterium in creating helium is an important factor. This destruction of deuterium is what yields the sensitivity of n_d to $\Omega_{\rm b}h^2$ so that it becomes a useful diagnostic.

The peak abundance of deuterium is not as sensitive to $\Omega_{\rm b}h^2$ as Eq. (24) implies, largely because adjustments in this cosmological parameter are accompanied by adjustments to the temperature/redshift for neutron freeze-out.

Plot: Schramm and Wagoner History of Light Element Abundances

- Observe that the $n_d = n_n$ *crossover* occurs at $T \sim 8 \times 10^8\,$K, i.e. at time $t \approx 3$ minutes. *This identifies the age of the Big Bang nucleosynthetic epoch.*