3 Recombination Era

- Consider the **dissociation/recombination** atomic equilibrium between **C & O, pp. 1181-2**

$$e^- + p \;\leftrightarrow\; H + \gamma \quad . \tag{32}$$

Occupation numbers for this equilibrium are $N \ll 1$, so that each species (other than photons) behaves like a classical gas: $N = \exp\{(\mu - E)/kT\}$. Then the number densities are prescribed by

$$n \;=\; \frac{g}{(2\pi\hbar)^3} \int d^3p \, e^{(\mu-E)/kT} \;=\; g \left(\frac{m\, kT}{2\pi\hbar^2} \right)^{3/2} e^{(\mu - mc^2)/kT} \tag{33}$$

for non-relativistic particles with energy $E = mc^2 + p^2/(2m)$. Here, the chemical potential is μ, and equilibrium implies $\mu(e) + \mu(p) = \mu(H)$ since $\mu(\gamma) = 0$ as photons have zero rest mass.

∗ Also, g is the number of spin states so that $g_e = 2 = g_p$ and $g_H = 4$.

The binding energy associated with this equilibrium is

$$E_B \;\equiv\; (m_e + m_p - m_H)c^2 \;=\; \chi_H \;\equiv\; 13.6\,\text{eV} \quad . \tag{34}$$

Evaluating Eq. (33) for electrons, protons and neutral hydrogen, and multiplying/dividing as appropriate gives

$$\frac{n_e n_p}{n_H\,(n_p + n_H)} \;\equiv\; \frac{x^2}{1-x} \;=\; \left(\frac{m\, kT}{2\pi\hbar^2} \right)^{3/2} \frac{e^{-E_B/kT}}{(n_p + n_H)} \quad , \tag{35}$$

where

$$x \;=\; \frac{n_e}{n_p + n_H} \;=\; \frac{n_p}{n_p + n_H} \tag{36}$$

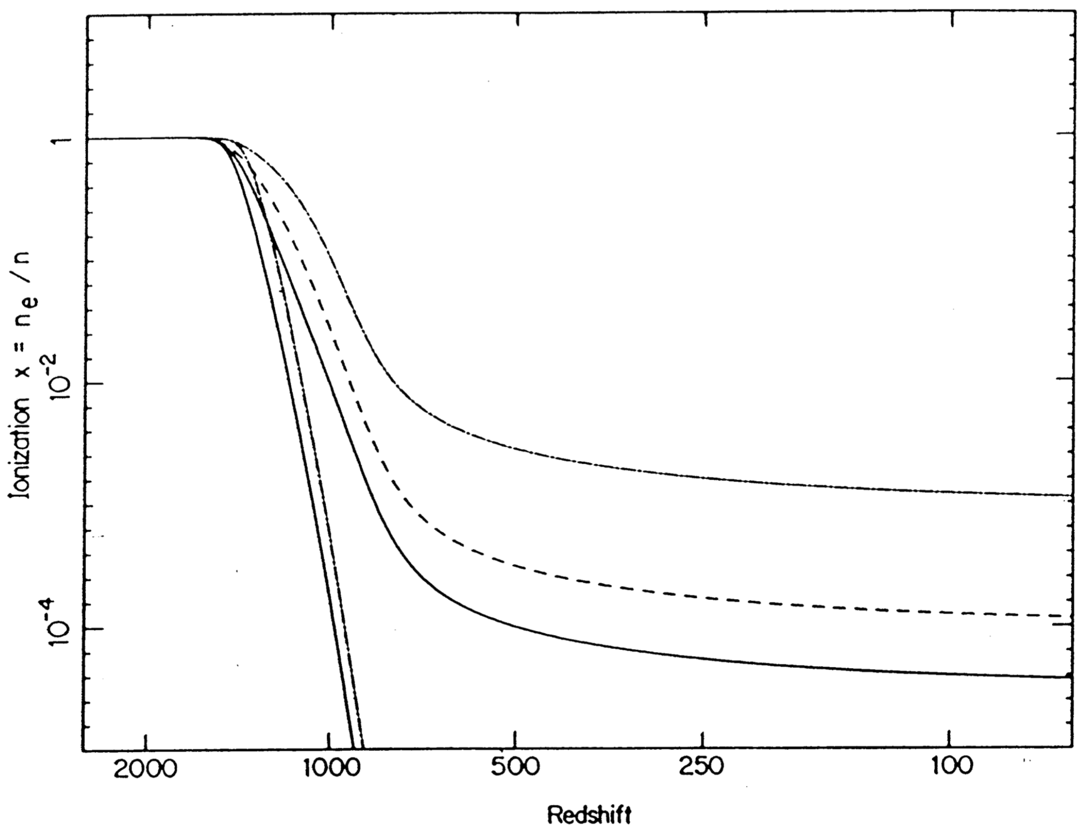
is the ionization fraction. In general, $x = x(z)$.

Eq. (35) is the **Saha equation** for recombination thermal equilibrium, and is widely employed in the treatment of star formation, stellar structure and the influences of stars on their gaseous environs.

Plot: Ionization Fraction in Early Universe

We will use this as the basis of our recombination pedagogy, noting that in practice, it has to be modified to account for the dynamic, evolving Universe.

Ionization Fraction at Recombination



$\Omega h^2 = \Omega_b h^2 = 1$

$\Omega h^2 = \Omega_b h^2 = 0.1$

- Evolution of ionization: Fig. 6.8 of Peebles. Steeply falling curves are Saha equilibrium computations. The curves that flatten are for the fuller dynamic calculation of Peebles (1969).
- Solid curves are for $\Omega h^2 = \Omega_b h^2 = 0.1$; dashed curve is for $\Omega h^2 = 1$ and $\Omega_b h^2 = 0.1$; dot-dashed curves are for $\Omega h^2 = 1$ and $\Omega_b h^2 = 1$. The plateaux at more recent times are characteristic of "freeze-out" eras in phase transitions.

• Observing that matter and radiation are still thermally coupled at recombination, we can set $T = 2.726(1+z)\,\mathrm{K}$ and ignore the contribution of helium to the computation. Then, for baryonic number density $n = 1.12 \times 10^{-5}(\Omega_b h^2)\,(1+z)^3\,\mathrm{cm}^{-3}$ with $h = H_0/[100\,\mathrm{km/sec/Mpc}]$, the Saha equation can be written in logarithmic space as

$$\log_e\left[\frac{x^2}{1-x}\right] \;\approx\; 48.3 - \frac{57940}{1+z} - \log_e\left\{\Omega_b h^2\,(1+z)^{3/2}\right\} \quad . \tag{37}$$

The first two terms on the RHS dominate the determination of the solution for z. Hence, with $\Omega_b h^2 = 0.0227$ from WMAP, setting $x = 0.5$ then gives a **redshift for recombination** of

$$\boxed{z_{\rm rec} \;\approx\; 1380\;,} \tag{38}$$

corresponding to a recombination temperature $T_{\rm rec} \sim 3,800\,\mathrm{K}$ that is much lower than the $10^4\,\mathrm{K}$ value that would signal full ionization.

$*$ This redshift is higher than the commonly cited value of $z_{\rm rec,ls} \sim 1,100$ by the WMAP Collaboration, which is actually a combination of recombination and photon-electron decoupling (last scattering) epochs.

4 Redshift of Last Scattering

The beautiful Planck spectrum obtained by COBE implies an upper limit to the redshift $z_{l.s}$ of **last scattering**, i.e. when the Thomson optical depth $\tau_{es}(z)$ for scatterings involving *ionized electrons* (by now of small abundance) is unity between redshifts $z = 0$ and $z = z_{l.s}$. As the universe becomes progressively more neutral, the scattering probability drops.

$$\tau_{es}(z) \;=\; \langle n_e \sigma_{\rm T} R\rangle \;\approx\; 0.12\,\Omega_b\,h(1+z)^2 x(z) \tag{39}$$

for $h = H_0/[100\,\mathrm{km/sec/Mpc}]$. Here $x(z)$ is the **ionization fraction** in the universe, the spatial scale is $R \approx ct_H(z)$ with $H(z) \propto h$.

• Observe that $\tau_{es} \gtrsim 10^3$ at recombination, and rapidly drops as the universe becomes more tenuous. Note also that the redshift dependences $n_e \propto 1/V \propto (1+z)^3$ and $R \propto 1/(1+z)$ lead to the $(1+z)^2$ factor in Eq. (39).

• Energy exchange between e^- and photons arises when they are not of equal temperature, and is called the **Sunyaev-Zel'dovich** effect. The exchange is given by the formula for loss of radiation energy ($U \propto T^4$):

$$\frac{\Delta U}{U} = 4 \int \frac{(kT_e - kT_\gamma)}{m_e c^2} \, d\tau_{es} \quad . \tag{40}$$

For $z \gtrsim z_{l.s}$, $kT_e = kT_\gamma$ so that $\Delta U/U = 0$, i.e. scattering perpetuates equipartition. After the epoch of last scattering, when $z \ll z_{l.s}$,

$$T_e = T_{e,0}(1+z)^2 \ll T_\gamma = T_{\gamma,0}(1+z) \tag{41}$$

where $T_{\gamma,0} = 2.726\,\mathrm{K}$. Then T_e can be neglected and the change of variables to z can be effected using Eq. (39), so that

$$\frac{\Delta U}{U} = -4\frac{kT_{\gamma,0}}{m_e c^2} \tau_{es}(z_{l.s}) \approx -1.5 \times 10^{-9} \, \Omega_b \, h z_{l.s}^2 \, x(z_{l.s}) \quad . \tag{42}$$

The COBE observations established the bounds

$$\frac{|\Delta T|}{T} \lesssim 6 \times 10^{-6} \quad \Rightarrow \frac{|\Delta U|}{U} = 4\frac{|\Delta T|}{T} \lesssim 2 \times 10^{-5} \quad , \tag{43}$$

which can be applied to Eq. (42) using the value $\Omega_b \sim 0.046$ from light element nucleosynthesis. The resulting COBE bound is

$$\boxed{z_{l.s} \sim 800 \ .} \tag{44}$$

Including neutrals yields a more accurate determination of $\tau_{es}(z)$, and it then becomes a much stronger function of z around $z \sim z_{l.s}$; the reduction of the free electron abundance yields modestly higher estimates of $z_{l.s}$.

• A complimentary consideration concerns the Sunyaev-Zel'dovich effect in the hot intracluster gas in galaxy clusters. This is at $T_e \sim 10^7\,\mathrm{K}$ from X-ray bremsstrahlung observations, and the absence of distortion of the CMB places an alternative and constraining bound on the mean free e^- density (i.e. ionization fraction) at redshifts $z \lesssim 1$, at least on localized n_e densities near such structures.

9. BIG BANG NUCLEOSYNTHESIS

Matthew Baring – Lecture Notes for ASTR 360, Spring 2025

1 Primordial Nucleosynthesis by the *pp* Chain

One of the key markers of the Big Bang model is primordial nucleosynthesis of low mass elements. It was well known that stellar nucleosynthesis models underproduced helium and other light elements relative to observed average abundances. Compact, high temperature epochs in the early universe are inevitable in a Big Bang scenario $\Rightarrow$ rampant nuclear burning.

• During the first few minutes of evolution of the universe, nucleosynthesis is prolific and succeeds in generating the primordial abundances of light elements that we observe today via a complex network of nuclear interactions.

Plot: Network of Primordial Nucleosynthetic Reactions

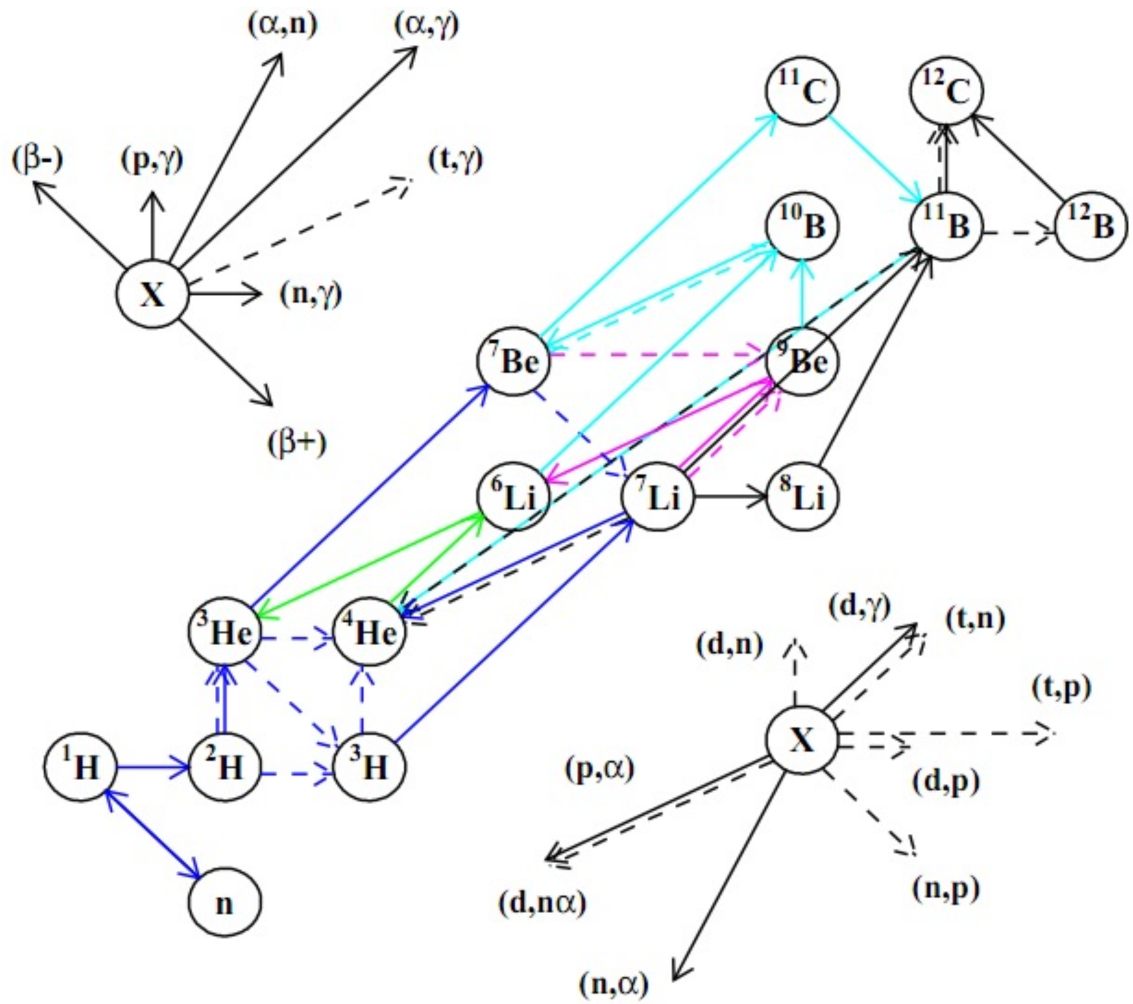
Numerical computations of light element abundances in an evolving early universe generally take into account many paths in this network up to isotopes of C, N and O. Time is insufficient to create heavier elements.

• At redshifts $z \gtrsim 10^{10}$, when $kT \gtrsim 3\,\text{MeV}$, photodissociation breaks up heavy elements into their constituent elementary particles: $n, p, \nu, \gamma, e^{\pm}$ are the predominant species. Subsequent to this, the ν background forms. The primary processes in this **proto-nuclear soup** are the relatively slow *weak interactions* that do not build nuclei:

$$\begin{aligned} e^- + p &\leftrightarrow n + \nu \\ \bar{\nu} + p &\leftrightarrow n + e^+ \\ n &\leftrightarrow p + e^- + \bar{\nu}\ . \end{aligned} \tag{1}$$

As the Universe ages, time is sufficient for fusion to become prolific.

Primordial Nucleosynthesis Network



Adapted from depiction in Fig. 2 of Schramm & Wagoner (1977)

- Reduced nuclear synthesis network displaying important reactions for D, He, Li, Be and beyond. Reaction types are indicated in the upper left and lower right.
- From Coc et al., ApJ 744, 158 (2012) [arXiv: 1107.1117], Fig. 22 therein.

1.1 Neutrino Decoupling

Early in the era of primordial nucleosynthesis, once quarks have morphed into hadrons, the first species to **decouple** or **freeze out** are the neutrinos, principally due to their weakly interacting character. Electron-positron pairs still abound so that the dominant weak interactions are

C & O, pp. 1172-3

$$e^- + e^+ \leftrightarrow \nu_e + \overline{\nu}_e \ , \quad e^\pm + \nu_e \leftrightarrow e^\pm + \nu_e \ , \quad e^\pm + \overline{\nu}_e \leftrightarrow e^\pm + \overline{\nu}_e \quad . \quad (2)$$

The cross sections generally scale as $\sigma \sim 3 \times 10^{-45} (E_\nu / m_e c^2)^2$ cm^2. The neutrinos are relativistic in this epoch, so $E_\nu \sim 3kT \approx 3k \times 2.73(1+z)$ since their temperature will be tied to radiation then.

The pairs are in chemical equilibrium with the neutrinos and photons, and so their density traces that of the photons, i.e. $n_\pm \sim n_\gamma \sim 410(1+z)^3 \, \text{cm}^{-3}$, remembering the present CMB photon density that was obtained from the Planck spectrum: $n_\gamma = 0.244 \, \Theta^3 / \lambda\!\!\!^{-3}$ for $\Theta = kT/m_e c^2$.

The timescale for the weak interactions is $t_{\rm w} \sim (\sigma n_\pm c)^{-1}$, with the pair density scaling as $n_\pm \propto a^{-3} \propto (1+z)^3$. Accordingly, this **neutrino decoupling timescale** has a strong dependence on redshift $z = 10^{10} z_{10}$:

$$t_{\rm w} \approx \frac{1}{n_\pm \sigma c} \approx \frac{1.4 \times 10^{49}}{(1+z)^5} \, \text{sec} \sim \frac{0.14}{(z_{10})^5} \, \text{sec} \quad . \quad (3)$$

Observe that since $\rho_c / m_p \approx 5.7 \times 10^{-6}$ protons per cc (now, with $\rho_c = 3H_0^2 / 8\pi G$) for $h = 0.71$, the number density of pairs in this early epoch exceeds that for baryons by a factor of $410/[5.7 \times 10^{-6}] = 7.2 \times 10^7$.

- In this early epoch that is dynamically radiation-dominated, the age $t(z)$ of the universe is given by

$$H_0 t(z) = \int_z^\infty \frac{dz'}{(1+z')\, E(z')} \quad , \quad E(z') \approx \sqrt{\Omega_{\rm rad}} \, (1+z')^2 \quad , \quad (4)$$

where the mass/energy evolution factor $E(z)$ simplifies considerably. Thus,

$$t(z) \approx \frac{1}{2H_0 \sqrt{\Omega_{\rm rad}}} \frac{1}{(1+z)^2} \sim \frac{0.31}{(z_{10})^2} \, \text{sec} \quad , \quad (5)$$

where $\Omega_{\rm rad} \sim 4.8 \times 10^{-5}$. The Universe ages more slowly as z drops than the weak interaction timescale grows, and thus decoupling does occur.