• Logic: gravity attracts, so decelerative solutions need to be balanced by a repulsive force. Since $\nabla^2\phi = 4\pi G(\rho + 3P)$ results from an attractive force, a $T_{00} < 0$ contribution to the energy-momentum tensor $T_{\mu\nu}$ is desired. The simplest path to this is to generalize the field equations to the form

$$\boxed{G_{\mu\nu} \;\equiv\; R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R \;=\; \frac{8\pi G}{c^4}\, T_{\mu\nu} - \Lambda g_{\mu\nu} \quad .} \tag{20}$$

Here Λ is the **cosmological constant**.

* $\Lambda > 0 \Rightarrow$ repulsion.
* $\Lambda =$ constant is consistent with the cosmological principle.

Friedmann's equation then generalizes to the form

$$(\dot{R})^2 - \frac{8\pi}{3}\, G \left[\rho + \frac{U_{\rm rad}}{c^2} \right] R^2 \;=\; -kc^2 + \frac{\Lambda}{3}\, R^2 \quad . \tag{21}$$

- Cosmologists also like to use the w parameter for each component, which is defined via an equation of state $P = w\rho c^2$, from which energy conservation derives $\rho \propto a^{-3(1+w)} \propto (1+z)^{3(1+w)}$. Thus, for a cosmological constant, $w = -1$, while $w = 0$ for cold normal matter, and $w = 1/3$ for radiation.

5.4 Luminosity Distance

The comoving volume element in the RW metric is (with $\chi \to \Theta$) **C & O, pp. 1212–5**

$$dV \;=\; 4\pi\Big[a_0 S_k(\chi)\Big]^2\, a_0 d\chi \quad , \tag{38}$$

and the redshift sets the luminosity at a different frequency. Hence, if the total "comoving" source luminosity is L, we can use the observed flux

$$\mathcal{F}_{obs} \;=\; \frac{L}{4\pi d_L^2} \tag{39}$$

(energy/time/area) to define the **luminosity distance** d_L.

• We know already that the solid angle scales as $\big[a_0 S_k(\Theta)\big]^2$. However, in addition, the photon energies are reduced in transit (e.g. optical $\to$ IR), leading to an extra $(1+z)$ factor reducing the observed flux. Also, the arrival rates are reduced by the same $(1+z)$ factor (time dilation), so that

$$\mathcal{F}_{obs} \;=\; \frac{L}{4\pi\big[a_0 S_k(\Theta)\big]^2 (1+z)^2} \quad , \tag{40}$$

implying that for $a_0 \to c/H_0$ (we now use real spatial units),

$$\boxed{d_L \;=\; \frac{c\,(1+z)}{H_0\sqrt{|\Omega-1|}}\; S_k(\Theta) \;\equiv\; (1+z)^2\, d_A \quad .} \tag{41}$$

Using Eq. (35) for the development angle, we arrive at

$$d_L \;=\; \frac{c\,(1+z)}{H_0\sqrt{|\Omega-1|}}\; S_k\left\{\sqrt{|\Omega-1|}\;\int_0^z \frac{dz'}{E(z')}\right\} \quad , \tag{42}$$

which clearly exhibits the property that $d_L \to cz/H_0$ as $z \to 0$, regardless of the geometry.

• The essential point is that **how distance is measured is actually dependent on the measurement technique** — using brightness or angular scale yields different determinations! In practice, d_L is more useful.

$*$ Observe that in the local universe limit, the angular diameter and luminosity distances coalesce! *This is independent of the cosmology.*

The flux-distance relation can also be converted to an *apparent magnitude* (m) – *redshift* relation (i.e., distance modulus):

$$m \;=\; -2.5 \log_{10} L + 5 \log_{10} d_L \quad . \tag{43}$$

Hence, for standard candles with fixed L, determining $m(z)$ probes the cosmology, i.e. measures q_0, Ω, etc. This is the principle of Type Ia supernova cosmology projects that we have discussed already.

Plot: SN1a apparent magnitude – redshift relation

These observational cosmology plots relating brightness to distance and therefore curvature of the universe are often known as **Hubble Diagrams**. Deviations from a linear dependence in such correlations lead to determinations of Ω_m and Ω_Λ, or the deceleration parameter q_0. Specifically, for our approximately flat universe with $\Omega \approx 1$ and $\Omega_{\rm rad} \ll 1$, one assumes the $k = 0$ portion of Eq. (34). Thus, $q_0 \approx \Omega_m/2 - \Omega_\Lambda$ and

$$d_L \;\approx\; \frac{c\,(1+z)}{H_0} \int_0^z \frac{dz'}{E(z')} \quad \text{for} \quad E(z) \;\approx\; \sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda} \quad . \tag{44}$$

It follows that

$$\frac{H_0}{c}\, \frac{d^2 d_L}{dz^2} \;\approx\; \frac{1}{2\,E(z)} + \frac{3\Omega_\Lambda}{2\,[E(z)]^3} \tag{45}$$

describes the magnitude-redshift relation curvature. Thus, while the $\Omega_m = 1\,, \Omega_\Lambda = 0$ Hubble diagram models possess some upward curvature, the curvature *increases* for our $\Omega_m = 0.27\,,\ \Omega_\Lambda = 0.73$ ($q_0 < 0$) universe.

∗ Two major competing groups, *Supernova Cosmology Project* (Perlmutter et al.) and *High-z Supernova Search team* (Reiss, Filippenko, et al.) arrived at commensurate results, leading to the 2011 Nobel Prize in Physics.

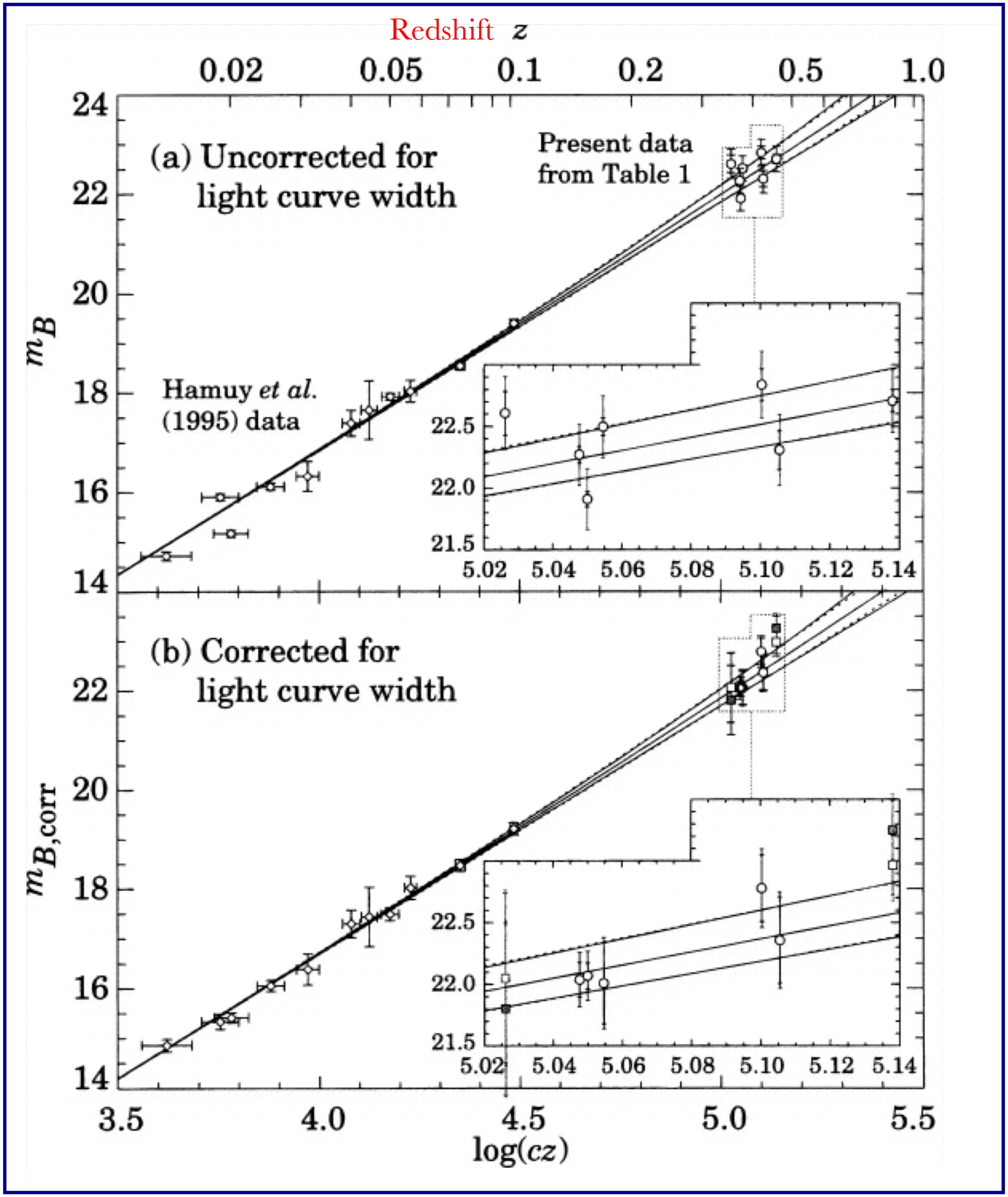
∗ SN Cosmology probes are most sensitive to differences between the matter and vacuum density parameters, i.e. pull versus push; radiation is not a player in the game because its current density is so small.

Plot: $\Omega_m - \Omega_\Lambda$ phase space for SN Ia probes

- Completely independent techniques, i.e. WMAP, Planck and the CMB mapping, and galaxy cluster determinations, later yielded confirmation of the global phase space parameters.

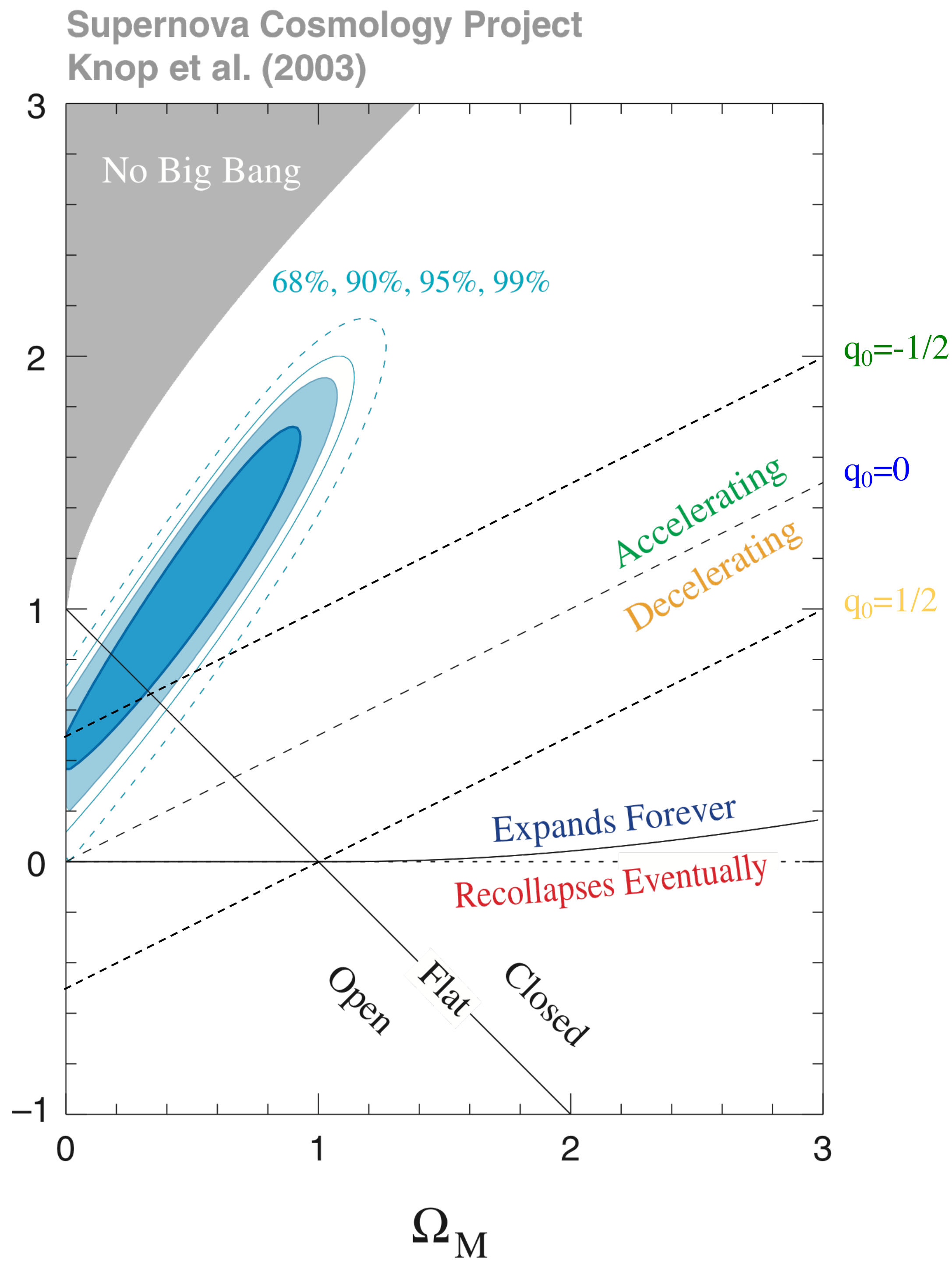
SN Ia Hubble Diagram

- Apparent magnitude-redshift diagram for SN Ia from Supernova Cosmology Project.
- Fig. 4 of Perlmutter* et al. ApJ 483, 565 (1997).
- The degree of upward curvature imposes the constraint $\Omega_\Lambda > \Omega_m$.
- *Nobel Prize in Physics 2011, along with Riess & Schmidt.



Ω Space for Supernova Cosmology

- Phase space for cosmic densities from SN Ia studies of Supernova Cosmology Project with HST.
- Knop et al., ApJ 598, 102 (2003), Fig. 7 therein.
- WMAP has reduced the constraining confidence contours substantially.



6 Summary of Useful Cosmology Equations

- The full Friedmann Equation for FRW universes is

$$\boxed{(\dot{R})^2 - \frac{8\pi}{3}\, G \left[\rho + \frac{U_{\rm rad}}{c^2}\right] R^2 \;=\; -kc^2 + \frac{\Lambda}{3}\, R^2 \quad .} \tag{46}$$

Setting $\rho = \Omega_m \rho_c$, $U_{\rm rad}/c^2 = \Omega_{\rm rad}\rho_c$ for $\rho_c = 3H_0^2/(8\pi G)$, it integrates to give the **age function** for the universe: **C & O, pp. 1192–3**

$$t \;=\; \frac{1}{H_0}\int_0^a \frac{d\Gamma}{\sqrt{\Omega_m/\Gamma + \Omega_{\rm rad}/\Gamma^2 + \Omega_\Lambda\Gamma^2 + (1-\Omega)}} \quad , \tag{47}$$

where $\Omega = \Omega_m + \Omega_{\rm rad} + \Omega_\Lambda$. Often $\Omega_\Lambda = \Lambda/(3H_0^2)$ is written Ω_v for the vacuum contribution.

- In terms of redshift z, this age function can be written

$$t(z) \;=\; \frac{1}{H_0}\int_z^\infty \frac{dz'}{(1+z')\, E(z')} \quad , \tag{48}$$

where

$$E(z) \;=\; \sqrt{\Omega_m(1+z)^3 + \Omega_{\rm rad}(1+z)^4 + \Omega_\Lambda + (1-\Omega)\,(1+z)^2} \quad . \tag{49}$$

- The **luminosity distance** d_L and the **angular diameter distance** d_A satisfy

$$d_L \;=\; \frac{c\,(1+z)}{H_0\sqrt{|\Omega-1|}}\, S_k(\Theta) \;\equiv\; (1+z)^2\, d_A \quad , \tag{50}$$

for

$$S_k(\Theta) \;=\; \begin{cases} \sin\Theta, & k=1, \\ \Theta, & k=0, \\ \sinh\Theta, & k=-1 \;. \end{cases} \tag{51}$$

and a **development angle**

$$\Theta \;=\; \sqrt{|\Omega-1|}\,\int_0^z \frac{dz'}{E(z')} \quad . \tag{52}$$

8. COSMIC BACKGROUND RADIATION

Matthew Baring – Lecture Notes for ASTR 360, Spring 2025

1 Cosmic Microwave Background

The discovery of the Cosmic Microwave Background radiation (CMB) was the watershed vindication of the Big Bang theory which originated with the work of **George Gamow** in 1946-8. It was not a serendipitous result, but rather an anticipated and sought-after proof.

C & O, Sec. 29.2

• Alternative proposals for steady-state universes have not been vindicated, and were questioned even prior to the CMB discovery.

[*Reading Assignment: Steady-State Alternative Models, C&O p. 1163-4*]

1.1 Anticipation

• A small universe at early times necessarily implies high density ρ and high temperature T. Adiabatic expansion demarcates two distinct cases:

C & O, pp. 1164-5

$*$ 1. For radiation-dominated epochs, $kT \propto P \propto \rho^{\gamma-1} = \rho^{1/3} \propto 1/R$, since the equation of state is relativistic ($\gamma = 4/3$ is the adiabatic index);

$*$ 2. For matter-dominated epochs, $kT \propto P \propto \rho^{\gamma-1} = \rho^{2/3} \propto 1/R^2$, since the equation of state is non-relativistic ($\gamma = 5/3$ is the adiabatic index).

• These later epochs generate more rapid adiabatic cooling of the universe, to some present temperature, *unknown before 1965.*

The demarcation of these two epochs occurs when $\Omega_m(1+z)^3 \sim \Omega_{\rm rad}(1+z)^4$ in the $E(z)$ function, i.e. for a redshift of $z \sim \Omega_m/\Omega_{\rm rad} \sim 3,170$ using current WMAP or Planck data.

Plot: Density Evolution in a Two-Component Universe

$*$ The impact of a cosmological constant modifies this global picture of the thermal history of the universe only near the present epoch (i.e., $z \lesssim 1$), and so has only minor influence on the assertions below.

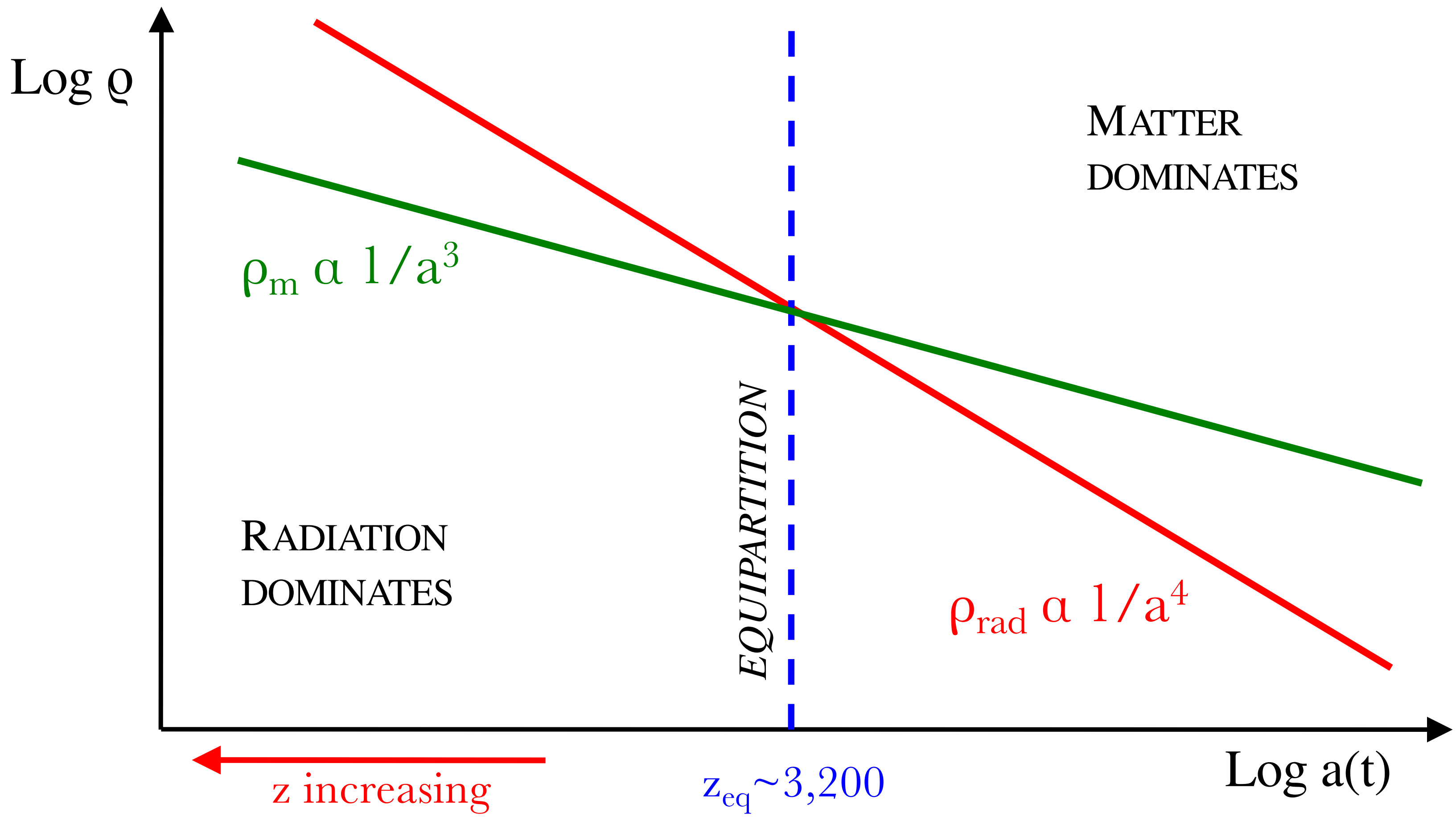
• Radiation and matter couple through:

- pair creation $\gamma\gamma \leftrightarrow e^+e^-$ when $T \gtrsim 10^{10}\,$K ($\Rightarrow z \sim 3 \times 10^9$);
- nuclear interaction when $T \gtrsim 10^9\,$K ($\Rightarrow z \sim 3 \times 10^8$);
- ionization/recombination at $T \gtrsim 4000\,$K ($\Rightarrow z \sim 10^3$);
- Compton scattering until $\tau_{\rm T} \lesssim 1$ ($\Rightarrow z \sim 800$).

At early times, a nuclear/pair "equilibrium" is pervasive, setting the scene for many photons, i.e. gamma-rays, coupled to nuclear reactions. At later times, the system is atomic in character, and when $T \lesssim 4000\,$K, the universe has passed the **decoupling** or **recombination** era.

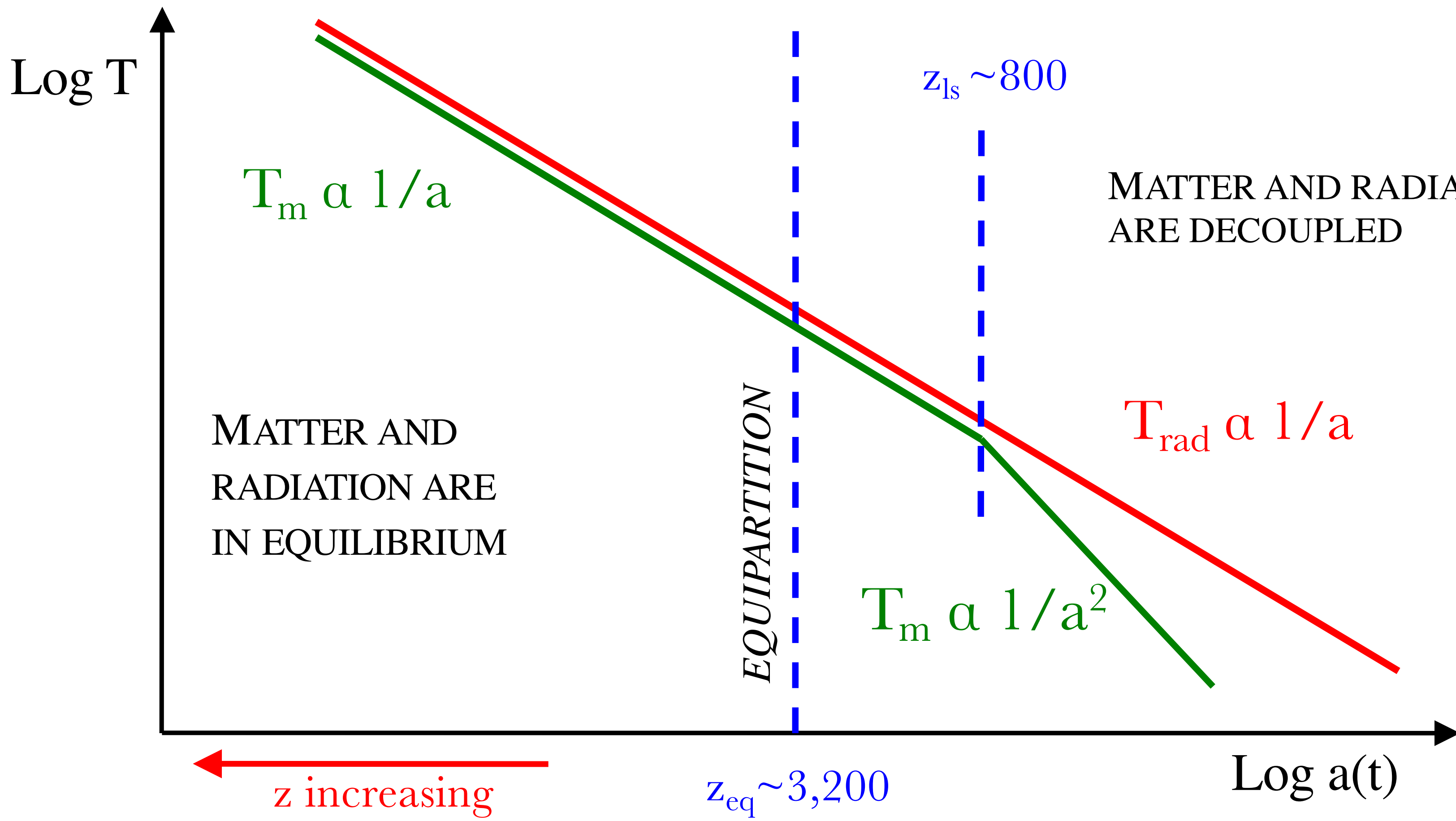
Plot: Temperature Evolution in a Two-Component Universe

Matter + Radiation Density Evolution



- Evolution of matter ρ_m and radiation $\rho_{rad}=U_{rad}/c^2$ density evolution with redshift z in a Universe where they are *always decoupled*.
- Equipartition is realized at z_{eq}~3,200, prior to which radiation controls the dynamic evolution of the Universe.

Matter + Radiation Temperature Evolution



- Evolution of matter T_m and radiation T_{rad} temperature evolution with redshift z in a Universe where they are decoupled after $z_{ls} < z_{rec}$.
- Thomson scattering transparency is realized at z_{ls}~800, prior to which matter and radiation are in energy equilibrium in the evolving Universe.