4 The Cosmological Constant

C & O, pp. 1190–1

- Einstein's theory came out before we had knowledge of the Hubble expansion. The prevailing view in 1916 was of a steady-state universe, for which we would ascribe $H_0 = 0$ in our present description.

The field equations do not exhibit $H_0 \approx 0$ solutions for most of parameter space. This led Einstein to propose a simple way to effect such behavior in model steady-state universes.

- Logic: gravity attracts, so decelerative solutions need to be balanced by a repulsive force. Since $\nabla^2\phi = 4\pi G(\rho + 3P)$ results from an attractive force, a $T_{00} < 0$ contribution to the energy-momentum tensor $T_{\mu\nu}$ is desired. The simplest path to this is to generalize the field equations to the form

$$\boxed{G_{\mu\nu} \;\equiv\; R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R \;=\; \frac{8\pi G}{c^4}\, T_{\mu\nu} - \Lambda g_{\mu\nu} \quad .} \tag{20}$$

Here Λ is the **cosmological constant**.

- $\Lambda > 0 \Rightarrow$ repulsion.
- $\Lambda =$ constant is consistent with the cosmological principle.

Friedmann's equation then generalizes to the form

$$(\dot{R})^2 - \frac{8\pi}{3}\, G \left[\rho + \frac{U_{\rm rad}}{c^2} \right] R^2 \;=\; -kc^2 + \frac{\Lambda}{3}\, R^2 \quad . \tag{21}$$

Thus, it is easy to discern that when the Λ term dominates, $\dot{a} \propto a$, i.e.

$$\frac{1}{a}\,\frac{da}{dt} \;=\; \pm\frac{1}{t_\Lambda} \;=\; \text{const.} \tag{22}$$

for $t_\Lambda = \sqrt{3/\Lambda}$, which leads to the exponential solution

$$a(t) \;\propto\; \exp\left\{ \pm\frac{t}{t_\Lambda} \right\} \quad . \tag{23}$$

The negative exponent corresponds to the growth of density perturbations in forming large scale structure (to be studied in due course).

For larger scale (cosmological) evolution, as the universe is expanding, the *positive exponent is pertinent*, resulting in the **de-Sitter solution**, which is realized when $\Lambda \gtrsim 8\pi G\rho$.

- The integration of Friedmann's equation parallels previous developments:

C & O, pp. 1192–3

$$\boxed{t \; = \; \frac{1}{H_0}\int_0^a \frac{d\Gamma}{\sqrt{\Omega_m/\Gamma + \Omega_{\rm rad}/\Gamma^2 + \Omega_\Lambda\Gamma^2 + (1-\Omega)}} \quad ,} \tag{24}$$

where $\Omega = \Omega_m + \Omega_{\rm rad} + \Omega_\Lambda$. Often $\Omega_\Lambda = \Lambda/(3H_0^2)$ is written Ω_v for the **vacuum contribution** to the universe's energy density. In general, this integral must be evaluated numerically.

Observe that the cosmological or vacuum term has a density that is independent of the scale parameter, i.e. constant in time, so that *eventually it wins out* over matter and radiation, whose densities decline with expansion.

$*$ If $\Lambda < 0$, recollapse is inevitable, either forced by Λ in low density universes at large a, or by $\Omega_m > 1$.

$*$ If $\Lambda > 0$ but $\Omega_m < 1$, then the model expands to infinity.

$*$ If $\Lambda > 0$ but $\Omega_m > 1$, then collapse can be avoided provided $\Omega_\Lambda > f(\Omega_m)$, where f is a cubic function (we won't worry about the details).

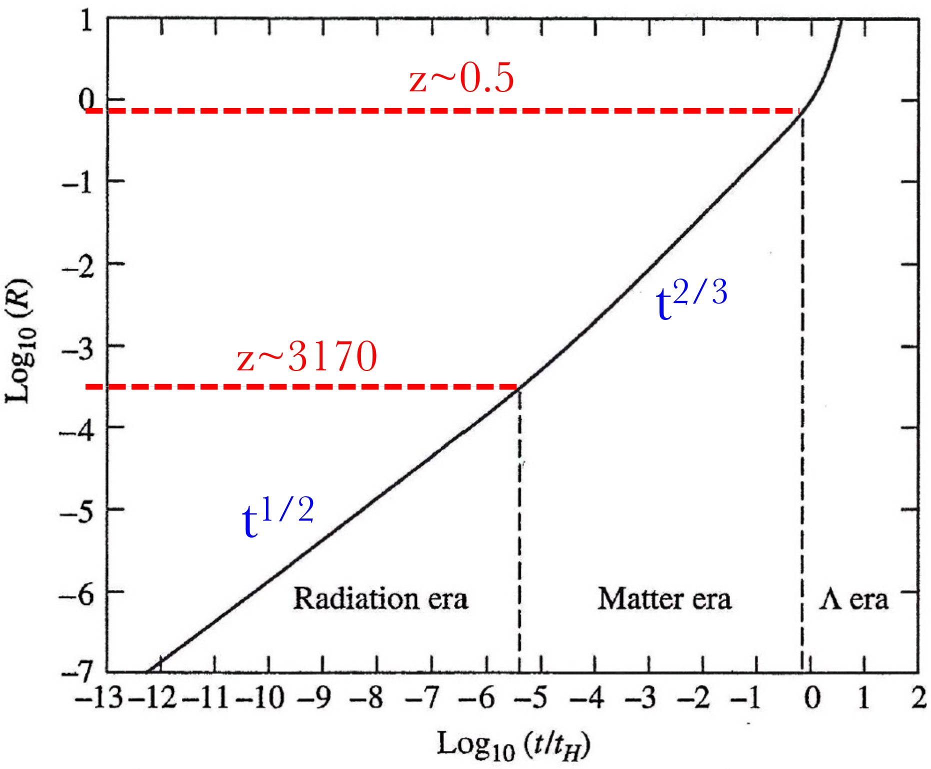
Plot: Expansion for a WMAP Universe

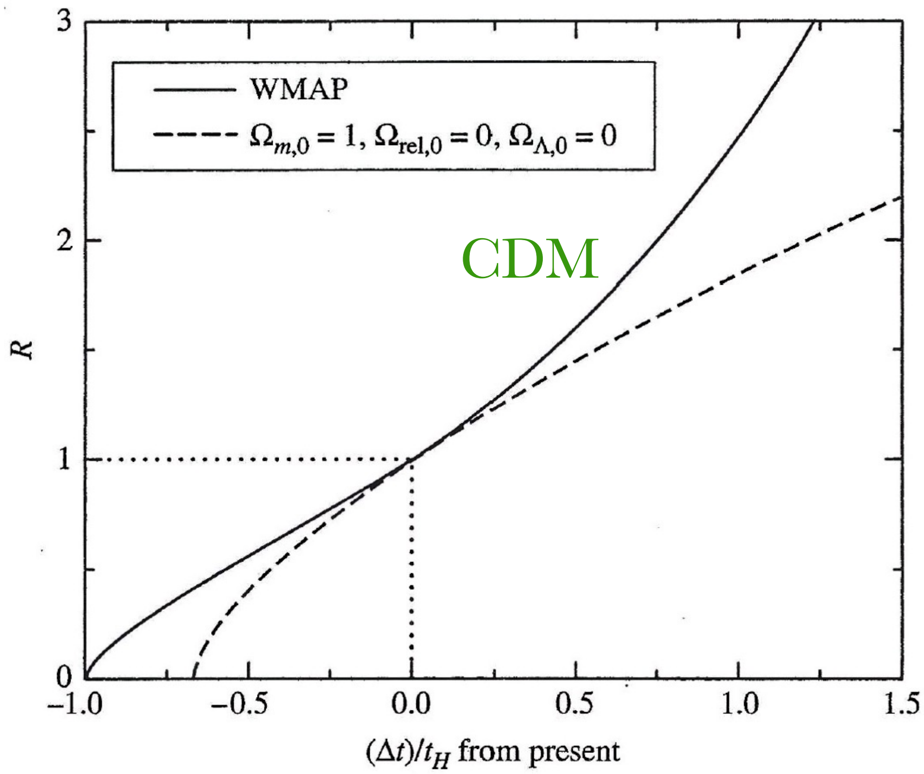
The flat, $k = 0$ case for $\Omega_{\rm rad} = 0$ is called the *Einstein-de Sitter* model.

- Back to Einstein's original premise, if we set $H_0 \to 0$ to mimic the perception at his time, then Friedmann's equation admits $\dot{a} = 0$ solutions of constant density and $k = {\rm sign}\Lambda$, constituting positive curvature for $\Lambda > 0$. This generates a **static universe**, as desired by Einstein. Upon Hubble's discovery of expansion, Einstein labelled his cosmological constant hypothesis *his biggest blunder.* Perhaps not!

- Cosmologists also like to use the w parameter for each component, which is defined via an equation of state $P = w\rho c^2$, from which energy conservation derives $\rho \propto a^{-3(1+w)} \propto (1+z)^{3(1+w)}$. Thus, for a cosmological constant, $w = -1$, while $w = 0$ for cold normal matter, and $w = 1/3$ for radiation.

Expansion of a WMAP Universe





- *Left*: Scale factor R=a during a broad range of epochs. Recently ($z<0.5$), we have a vacuum-dominated epoch following a matter-dominated one.
- *Right*: Comparison of scale factor evolution for a WMAP CDM universe (solid curve) and a flat, matter-dominated universe (dashed curve).
- Figures 29.19 and 29.20 of Carroll & Ostlie *An Introduction to Modern Astrophysics*

5 Connecting to the Real World

It is expedient to express some of the key expansion equations in terms of salient observables. The key observable is the (spectroscopic) redshift z:

$$1 + z \;=\; \frac{a_0}{a(t)} \quad . \tag{25}$$

All pertinent integrals can be expressed as integrations over z.

5.1 Deceleration Parameter

The full version of Friedmann's equation in Eq. (21) is equivalent to

$$\left(\frac{\dot{a}}{a}\right)^2 \;=\; H_0^2 \left\{ \Omega_m (1+z)^3 + \Omega_{\rm rad} (1+z)^4 + \Omega_\Lambda + (1-\Omega)\,(1+z)^2 \right\} \quad , \tag{26}$$

where $\Omega = \Omega_m + \Omega_{\rm rad} + \Omega_\Lambda$ as before. The acceleration/deceleration can be obtained by differentiating Eq. (26) with respect to time (remember, $a_0 = 1$):

$$\begin{aligned} 2\,\ddot{a}\,\dot{a} \;&=\; \dot{a}\, H_0^2 a_0^2 \,\frac{d}{da} \left\{ \Omega_m \left(\frac{a_0}{a}\right) + \Omega_{\rm rad} \left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda \left(\frac{a}{a_0}\right)^2 + \text{const} \right\} \\ \Rightarrow\quad \frac{\ddot{a}}{a} \;&=\; H_0^2 \left\{ \Omega_\Lambda - \frac{\Omega_m}{2} \left(\frac{a_0}{a}\right)^3 - \Omega_{\rm rad} \left(\frac{a_0}{a}\right)^4 \right\} \qquad (27) \\ &=\; H_0^2 \left\{ \Omega_\Lambda - \frac{\Omega_m}{2} (1+z)^3 - \Omega_{\rm rad} (1+z)^4 \right\} \quad . \end{aligned}$$

Evaluating this at $z = 0$ and combining with Friedmann's equation yields

$$q_0 \;=\; -\frac{\ddot{a}_0\, a_0}{(\dot{a}_0)^2} \;=\; \frac{\Omega_m}{2} + \Omega_{\rm rad} - \Omega_\Lambda \tag{28}$$

as the general form of a quantity called the **deceleration parameter**. As a perturbation parameter about the local Hubble flow, before 1995, q_0 was highly sought after as an indicator of the level of closure of the universe. It essentially expresses the curvature in Hubble diagrams to leading order.

- Observe that if $\Omega_\Lambda > 0$ dominates, then $q_0 < 0$, *implying an accelerating universe,* a total surprise to the cosmology community in 1997.

5.2 Lookback Times

Since $1+z = a_0/a$, the Taylor series for the lookback time $t_0 - t$ is

$$a(t) \;=\; a_0 \left[1 - H_0(t_0-t) - \frac{q_0}{2}\, H_0^2\, (t_0-t)^2 \dots \right] \quad , \tag{29}$$

which can routinely be inverted to give $t_0 - t$ as a function of z. However, we already have this inversion as a solution integral for Friedmann's equation:

$$H_0 t(z) \;=\; H_0 \int_0^a \frac{da}{\dot{a}} \;\equiv\; \int_0^a \frac{da}{a\, E(z)} \;=\; \int_z^\infty \frac{dz'}{(1+z')\, E(z')} \quad , \tag{30}$$

where

$$E(z) \;=\; \sqrt{\Omega_m (1+z)^3 + \Omega_{\rm rad} (1+z)^4 + \Omega_\Lambda + (1-\Omega)\,(1+z)^2} \tag{31}$$

appears in the Eq. (26) version of Friedmann's equation. Note that as $z \to 0$, $E(z) \to 1$. Consequently,

$$t_0 - t(z) \;=\; \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z')\, E(z')} \quad , \tag{32}$$

Plot: Lookback time versus redshift

5.3 Angular Diameter Distance

C & O, pp. 1215–8

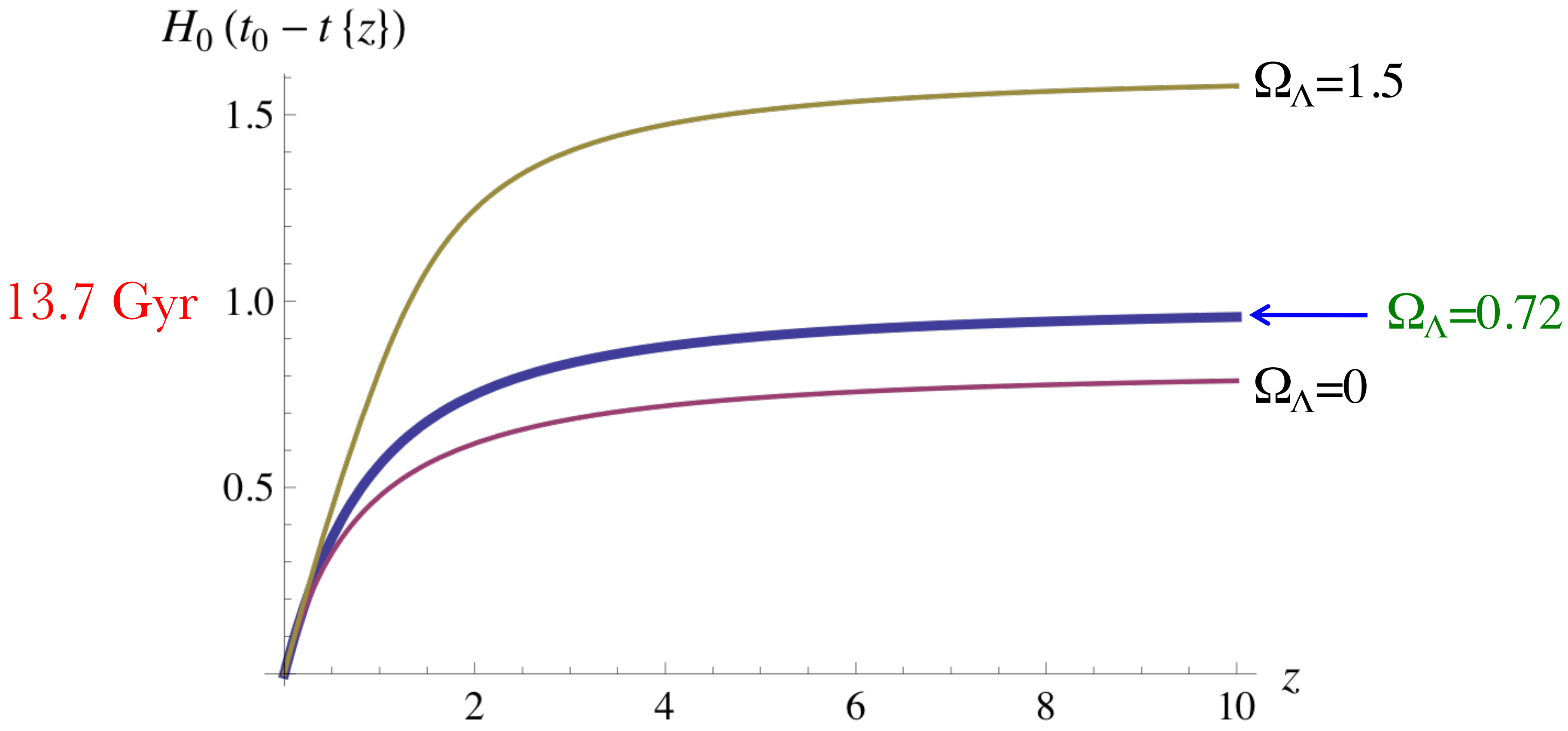
Because of the distortion from Euclidean geometry introduced by curved spacetime, the coupling between apparent angular size θ and distance d for an object of fixed size D deviates from $\theta = D/d \propto 1/z$. Considering

$$ds^2 \;=\; c^2\, dt^2 - a^2(t) \left\{ d\chi^2 + S_k^2(\chi)\, d\Omega \right\} \quad , \tag{33}$$

the RW line element, we now set $\chi \to \Theta$ to establish correspondence with the development angle Θ. The angular portion is captured via

$$S_k(\Theta) \;=\; \begin{cases} \sin\Theta, & k=1, \\ \Theta, & k=0, \\ \sinh\Theta, & k=-1 \;\; . \end{cases} \tag{34}$$

Lookback Times in Ω_m=0.28 Cosmologies



- Lookback time, scaled via $H_0(t-t_0[z])$, as a function of redshift back to around the re-ionization epoch. Radiation is insignificant (Ω_{rad}=0) and matter (including dark matter) is set at present density Ω_m=0.28.
- Different choices of the cosmological constant, labelled by Ω_Λ, are adopted, illustrating its influence on the total asymptotic age t_0 of the Universe as z becomes very large.

The physical area subtended by the source is then $a^2(t)\, S_k^2(\Theta)$, with $a(t)\, d\Theta$ being the radial distance element. Angular diameters therefore scale as $S_k(\Theta)/(1+z)$. For proper time elements $ds = 0$, then the radial scale (for $d\Omega = 0$) satisfies $d\Theta \propto dt/a = da/(\dot{a}\, a)$, so that

$$\Theta \; = \; H_0 \sqrt{|\Omega - 1|} \int_t^{t_0} \frac{a_0\, dt'}{a(t')} \; = \; \sqrt{|\Omega - 1|} \int_0^z \frac{dz'}{E(z')} \qquad (35)$$

expresses the development angle in terms of redshift z. Accordingly, we *define* the **angular diameter distance** via

$$\boxed{d_A \; \equiv \; \frac{c}{H_0 \sqrt{|\Omega - 1|}} \, \frac{S_k[\Theta(z)]}{1+z} \; = \; \frac{D}{\theta} \quad .} \qquad (36)$$

This gives the apparent behavior of angular scales of structures within the universe with redshift: angular size scales as $1/d_A$.

- As $z \to 0$, $\Theta \to z \sqrt{|\Omega - 1|}$, so that $d_A \to cz/H_0$. This then yields angular scales $\theta \propto H_0/z$ in the local (Euclidean) universe, as is expected.

Plot: Angular diameter distance versus redshift (from Peacock)

N.B. Since Θ remains finite as $z \to \infty$ [by inspection of Eq. (31)], then $d_A \propto (1+z)^{-1}$ i.e. angular scale $\theta \propto (1+z)$. Hence, placing a fixed object back at high redshift generates a large apparent angular size in the smaller universe, i.e. the object was looming over us when it emitted its light!

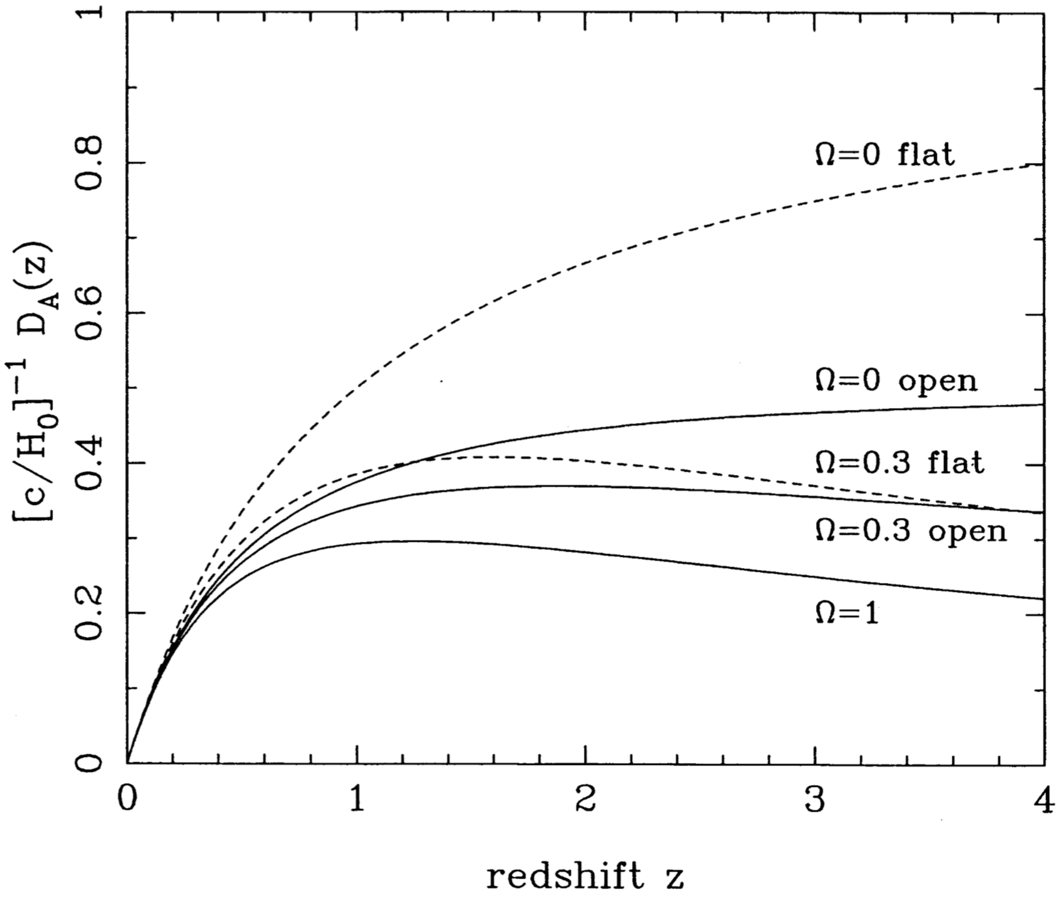
- For matter-dominated cosmologies, $\Omega_{\text{rad}} = 0 = \Omega_\Lambda$, the parametric form for $a(t)$ can be use to obtain an analytic form for $S_k(\Theta)$, and it can be shown that

$$d_A \; = \; \frac{c}{H_0} \, \frac{1}{q_0^2 (1+z)^2} \left\{ q_0 z + (q_0 - 1) \left[\sqrt{1 + 2 q_0 z} - 1 \right] \right\} \quad . \qquad (37)$$

Hence, in principle, observing variations in angular scales of known "standard size" objects leads to a determination of q_0. In practice, this is hard to do, as size variations hamper q_0 diagnostics.

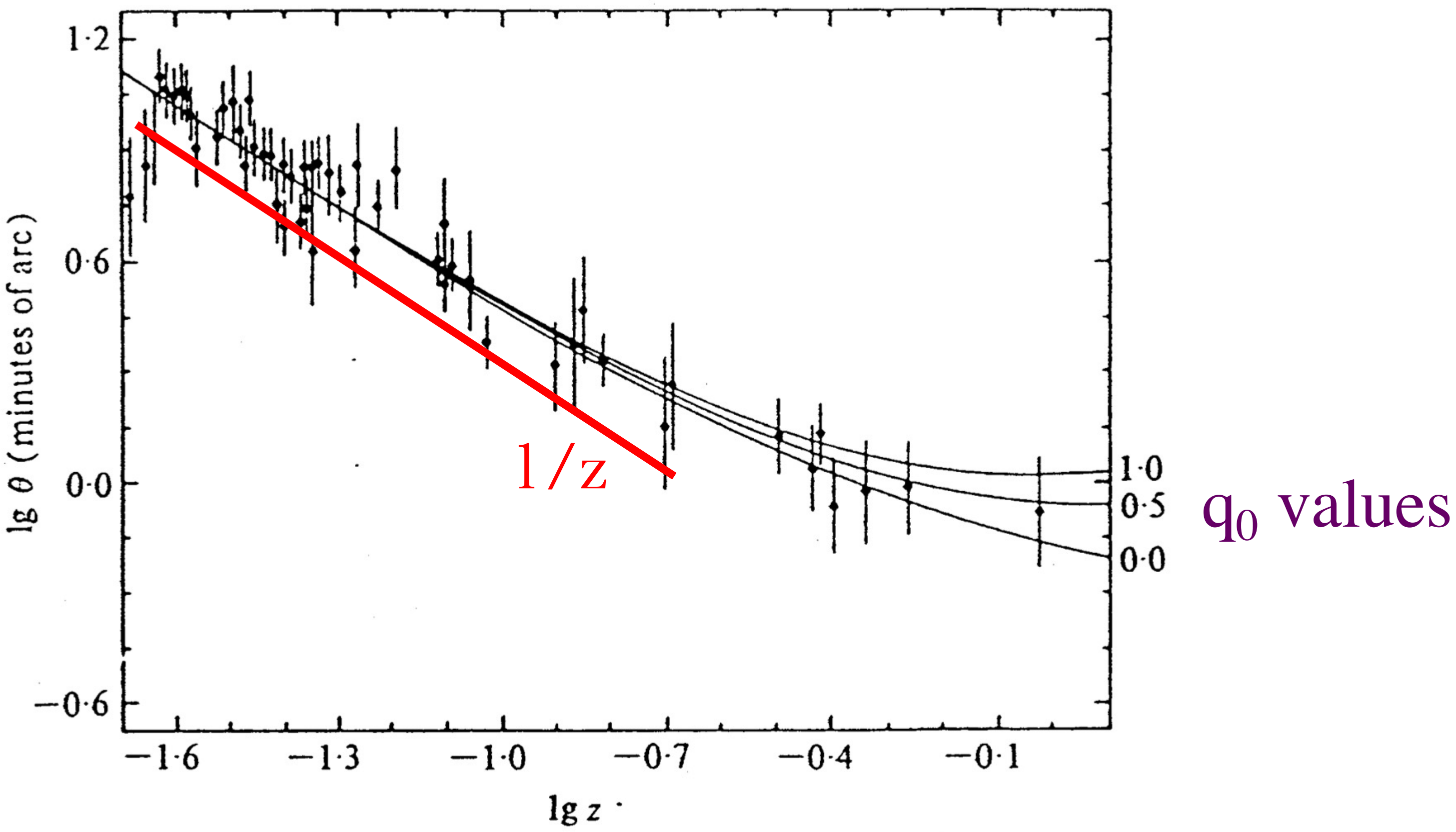
Plot: Angular diameter – redshift diagram for galaxy clusters

Angular Diameter Distance versus Redshift



- Scaled angular-diameter distance versus redshift for various cosmologies (Fig. 3.7 from Peacock *Cosmological Physics*). The solid curves show models with zero vacuum energy; the broken curves show flat models with $\Omega_m+\Omega_\Lambda=1$. In both cases, curves for $\Omega_m=0$, 0.3, 1 are depicted. Higher density results in lower diameter distances at high z due to greater gravitational focusing of light rays.

Angular Diameters for Galaxy Clusters



- Redshift dependence of angular diameters of clusters of galaxies (Fig. 7.8 from Rowan-Robinson *Cosmology*). The theoretical curves are for matter-dominated universes labelled by the deceleration parameter q_0 (=1/2 for flat cosmology).
- At low redshifts, the angular size scales as $\theta\ \alpha\ 1/z$ for all cosmologies .
- Uncertainty in diameter determination at $z\sim1$ precludes discrimination between open and closed universes => need for more precise tests of cosmological parameters.